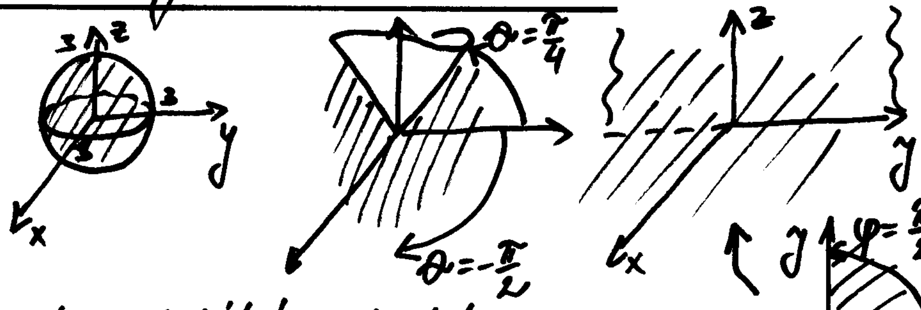
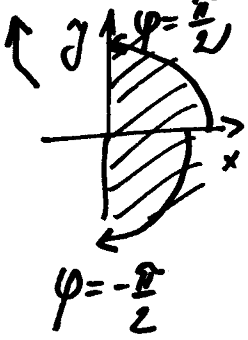
Spočtěte $\iint\limits_{\varphi} xy^2z\,dydz + z\,dxdz + (x^2+y^2)\,dxdy$, kde plocha φ je hranice tělesa $T=\{[x,y,z]\in\mathbb{R}^3 : x^2+y^2+z^2\le 9,\ z\le\sqrt{x^2+y^2},\ x\ge 0\}$. 1/2

Orientace plochy φ je dána vnější normálou.

Těleso T je část koule





Plocha φ je uzavřená plocha splňující podmínky pro použití Gaussovy-Ostrogradského věty:

$$\underbrace{\iint\limits_{\varphi} \vec{F}\,d\vec{S}}_{\text{plošný int. 2. druhu}} = \underbrace{\iiint\limits_{T} \operatorname{div}\vec{F}\,dxdydz}_{\text{trojný int.}}$$

$$\vec{F} = (xy^2z,\ z,\ x^2+y^2)$$

$$\operatorname{div}\vec{F} = \nabla\vec{F} = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(xy^2z,\ z,\ x^2+y^2) = y^2z + 0 + 0 = y^2z$$

$$\iint\limits_{\varphi}\vec{F}\,d\vec{S} = \iiint\limits_{T} y^2z\,dxdydz =$$

T je část koule, použijeme transformaci do sférických souřadnic

$$x = \rho\cos\varphi\cos\theta$$
$$y = \rho\sin\varphi\cos\theta$$
$$z = \rho\sin\theta$$
$$|\text{jakobián}| = \rho^2\cos\theta$$

Pozn.
VOLÍM SI VARIANTU, KDE ÚHEL θ JE ODKLON OD PŮDORYSNY
$\Downarrow$
$z = \rho\sin\theta$

$$0\le\rho\le 3$$
$$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$$
$$-\frac{\pi}{2}\le\theta\le\frac{\pi}{4}$$

$$= \int_0^3 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \underbrace{(\varrho \sin\varphi \cos\vartheta)^2}_{y^2} \cdot \underbrace{\varrho \sin\vartheta}_{z} \cdot \underbrace{\varrho^2 \cos\vartheta}_{|\text{jakobián}|} \, d\vartheta \right) d\varphi \right) d\varrho =$$

$$= \int_0^3 \varrho^5 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\varphi \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^3\vartheta \cdot \sin\vartheta \, d\vartheta \right) d\varphi \right) d\varrho = \left| \begin{array}{l} \text{substituce} \\ \cos\vartheta = t \\ -\sin\vartheta \, d\vartheta = 1\,dt \\ \text{přepočítáme meze} \\ -\frac{\pi}{2} \xrightarrow{\cos(-\frac{\pi}{2})} 0 \\ \frac{\pi}{4} \xrightarrow{\cos\frac{\pi}{4}} \frac{\sqrt{2}}{2} \end{array} \right| =$$

$$= \int_0^3 \varrho^5 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\varphi \left(\int_0^{\frac{\sqrt{2}}{2}} t^3 \cdot (-dt) \right) d\varphi \right) d\varrho =$$

$$= -\int_0^3 \varrho^5 \cdot \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\varphi \left[\frac{t^4}{4} \right]_0^{\frac{\sqrt{2}}{2}} d\varphi \right) d\varrho \qquad \underline{\underline{\sin^2\varphi = \frac{1-\cos 2\varphi}{2}}}$$

$$= -\left(\frac{\left(\frac{\sqrt{2}}{2}\right)^4}{4} - \frac{0^4}{4} \right) \int_0^3 \varrho^5 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 2\varphi}{2} \, d\varphi \right) d\varrho \qquad \underline{\underline{\int \cos 2\varphi \, d\varphi = \frac{\sin 2\varphi}{2}}}$$

$$= -\frac{1}{16} \cdot \frac{1}{2} \int_0^3 \varrho^5 \left[\varphi - \frac{\sin 2\varphi}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varrho = -\frac{1}{32} \left(\frac{\pi}{2} - \frac{\sin\left(2 \cdot \frac{\pi}{2}\right)}{2} - \left(-\frac{\pi}{2} - \frac{\sin\left(2 \cdot \left(-\frac{\pi}{2}\right)\right)}{2} \right) \right) \cdot$$

$$\cdot \int_0^3 \varrho^5 \, d\varrho = -\frac{1}{32} \cdot \pi \cdot \left[\frac{\varrho^6}{6} \right]_0^3 = -\frac{\pi}{32} \cdot \left(\frac{3^6}{6} - \frac{0^6}{6} \right) = -\frac{243\pi}{64} \doteq \underline{\underline{-11{,}9}}$$