DVOJNÝ INTEGRÁL (S TRANSFORMACÍ DO POLÁRNÍCH SOUŘADNIC)

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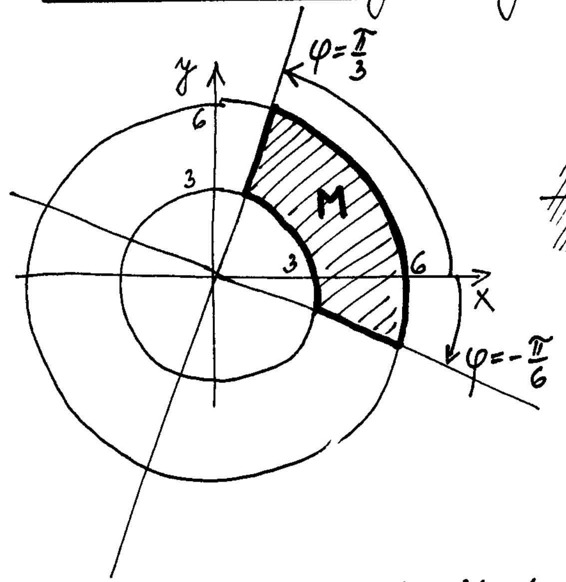
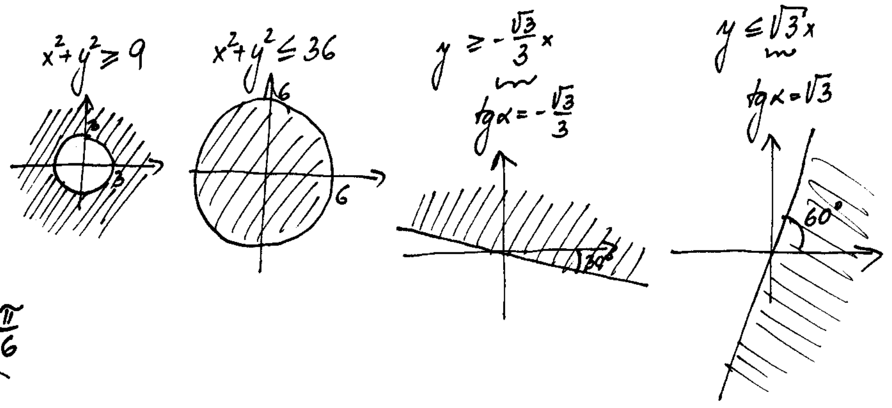
ÚM FSI VUT

Dvojným integrálem spočtěte hmotnost nehomogenní plochy M dané nerovnostmi $x^2+y^2\ge 9$, $x^2+y^2\le 36$, $y\ge -\frac{\sqrt{3}}{3}x$, $y\le\sqrt{3}\,x$.
Plošná hustota $\sigma(x,y)=2xy+100$.





Vzhledem k tvaru množiny M bude výhodné použít transformaci do polárních souřadnic:

$x=\rho\cos\varphi$ M: $3\le\rho\le 6$

$y=\rho\sin\varphi$ $-\frac{\pi}{6}\le\varphi\le\frac{\pi}{3}$

jakobián $=\rho$

$$m(M)=\iint_M\sigma(x,y)\,dx\,dy=\iint_M 2xy+100\,dx\,dy=\int_3^6\left(\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\Big(2\cdot\underbrace{\rho\cos\varphi}_{x}\cdot\underbrace{\rho\sin\varphi}_{y}+100\Big)\cdot\underbrace{\rho}_{\text{jakobián}}\,d\varphi\right)d\rho=$$

$$=\int_3^6\rho\left(\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\rho^2\cdot\underbrace{2\sin\varphi\cos\varphi}_{\sin 2\varphi}+100\,d\varphi\right)d\rho=\int_3^6\rho\left[\rho^2\left(-\frac{\cos 2\varphi}{2}\right)+100\varphi\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}d\rho=$$

$\int\sin 2\varphi\,d\varphi=\frac{-\cos 2\varphi}{2}$

$$=\int_3^6\rho\left(\rho^2\left(-\frac{\cos\left(2\cdot\frac{\pi}{3}\right)}{2}\right)+100\cdot\frac{\pi}{3}-\left(\rho^2\left(-\frac{\cos\left(2\cdot\left(-\frac{\pi}{6}\right)\right)}{2}\right)-100\cdot\frac{\pi}{6}\right)\right)d\rho=\int_3^6\rho^3\left(-\frac{-\frac{1}{2}}{2}+\frac{\frac{1}{2}}{2}\right)+\rho\cdot 100\cdot\frac{\pi}{2}\,d\rho=$$

$$=\left[\frac{1}{2}\frac{\rho^4}{4}+100\cdot\frac{\pi}{2}\cdot\frac{\rho^2}{2}\right]_3^6=\frac{1}{2}\cdot\frac{6^4}{4}+100\cdot\frac{\pi}{2}\cdot\frac{6^2}{2}-\left(\frac{1}{2}\cdot\frac{3^4}{4}+100\cdot\frac{\pi}{2}\cdot\frac{3^2}{2}\right)=\underline{\underline{\frac{135}{8}(9+40\pi)}}\doteq 2272{,}5$$