DVOJNÝ INTEGRÁL (S TRANSFORMACÍ DO POLÁRNÍCH SOUŘADNIC)

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ÚM FSI VUT

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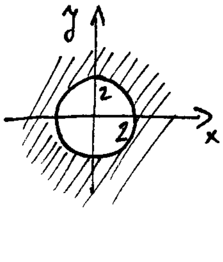
Spočtěte $\iint\limits_M 3x+2\,dx\,dy$, kde oblast M je dána nerovnostmi

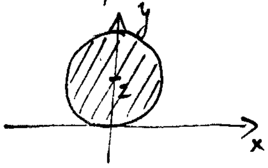
$x^2+y^2 \geq 4$, $x^2+y^2 \leq 4y$, $y \geq \sqrt{3}\,x$.



$x^2+y^2-4y \leq 0$

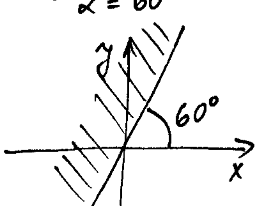
$x^2+(y-2)^2 \leq 4$

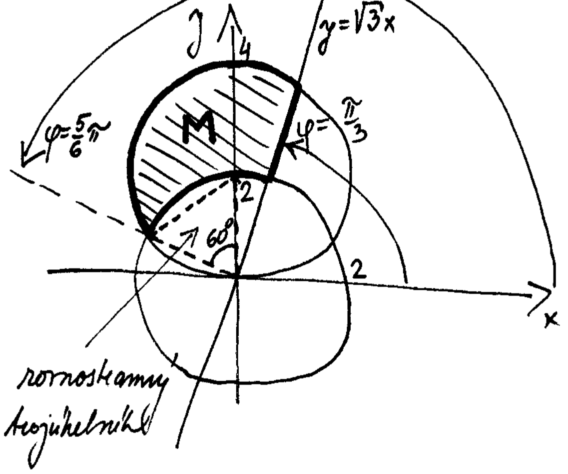
$S[0,2]$, $r=\sqrt{4}=2$



$\mathrm{tg}\,\alpha = \sqrt{3}$

$\alpha = 60°$





Vzhledem k tvaru množiny M bude výhodné použít transformaci do polárních souřadnic:

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

jakobián $= \rho$

$M:\quad 2 \leq \rho \overset{\circledast}{\leq} 4\sin\varphi$

$\frac{\pi}{3} \leq \varphi \leq \frac{5}{6}\pi$

$\circledast$ $x^2+y^2 \leq 4y$

$(\rho\cos\varphi)^2+(\rho\sin\varphi)^2 \leq 4\rho\sin\varphi$

$\rho^2\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1} \leq 4\rho\sin\varphi$

$\rho^2 \leq 4\rho\sin\varphi$

$\rho \leq 4\sin\varphi$

$$\iint\limits_M 3x+2\,dx\,dy = \int_{\frac{\pi}{3}}^{\frac{5}{6}\pi}\left(\int_2^{4\sin\varphi}(3\cdot\underbrace{\rho\cos\varphi}_{x}+2)\cdot\underbrace{\rho}_{\text{jakobián}}\,d\rho\right)d\varphi =$$

$$= \int_{\frac{\pi}{3}}^{\frac{5}{6}\pi}\left(\int_2^{4\sin\varphi}3\rho^2\cos\varphi+2\rho\,d\rho\right)d\varphi = \int_{\frac{\pi}{3}}^{\frac{5}{6}\pi}\left[3\cos\varphi\frac{\rho^3}{3}+\cancel{2}\frac{\rho^2}{\cancel{2}}\right]_2^{4\sin\varphi}d\varphi =$$

$$= \int_{\frac{\pi}{3}}^{\frac{5}{6}\pi}\cancel{3}\cos\varphi\frac{(4\sin\varphi)^3}{\cancel{3}}+(4\sin\varphi)^2-\left(\cancel{3}\cos\varphi\frac{2^3}{\cancel{3}}+2^2\right)d\varphi =$$

$$= \int_{\frac{\pi}{3}}^{\frac{5}{6}\pi}64\sin^3\varphi\cos\varphi+16\sin^2\varphi-8\cos\varphi-4\,d\varphi \;\overset{\sin^2\varphi=\frac{1-\cos 2\varphi}{2}}{=\!=\!=}$$

subs.
$\sin\varphi = t$
$\cos\varphi\,d\varphi = dt$
$\frac{\pi}{3} \xrightarrow{\sin\frac{\pi}{3}} \frac{\sqrt{3}}{2}$
$\frac{5}{6}\pi \xrightarrow{\sin\frac{5}{6}\pi} \frac{1}{2}$

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$$= 64\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} t^3\,dt + \overset{8}{\cancel{16}}\int_{\frac{\pi}{3}}^{\frac{5}{6}\pi} \frac{1-\cos 2\varphi}{\cancel{2}}\,d\varphi - 8\Big[\sin\varphi\Big]_{\frac{\pi}{3}}^{\frac{5}{6}\pi} - 4\Big[\varphi\Big]_{\frac{\pi}{3}}^{\frac{5}{6}\pi} \qquad \underline{\underline{\int \cos 2\varphi\,d\varphi = \frac{\sin 2\varphi}{2}}}$$

$$= \overset{16}{\cancel{64}}\left[\frac{t^4}{\cancel{4}}\right]_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} + 8\left[\varphi - \frac{\sin 2\varphi}{2}\right]_{\frac{\pi}{3}}^{\frac{5}{6}\pi} - 8\left(\sin\frac{5}{6}\pi - \sin\frac{\pi}{3}\right) - 4\left(\frac{5}{6}\pi - \frac{\pi}{3}\right) =$$

$$= 16\left(\left(\frac{1}{2}\right)^4 - \left(\frac{\sqrt{3}}{2}\right)^4\right) + 8\left(\frac{5}{6}\pi - \frac{\sin\left(2\cdot\frac{5}{6}\pi\right)}{2} - \left(\frac{\pi}{3} - \frac{\sin\left(2\cdot\frac{\pi}{3}\right)}{2}\right)\right) - 8\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) - 4\cdot\frac{\pi}{2} =$$

$$= (1-9) + 8\left(\frac{\pi}{2} - \frac{-\frac{\sqrt{3}}{2}}{2} + \frac{\frac{\sqrt{3}}{2}}{2}\right) - 4(1-\sqrt{3}) - 2\pi =$$

$$= -8 + 4(\pi + \sqrt{3}) - 4 + 4\sqrt{3} - 2\pi =$$

$$= \underline{\underline{-12 + 2\pi + 8\sqrt{3}}} \doteq 8{,}1$$