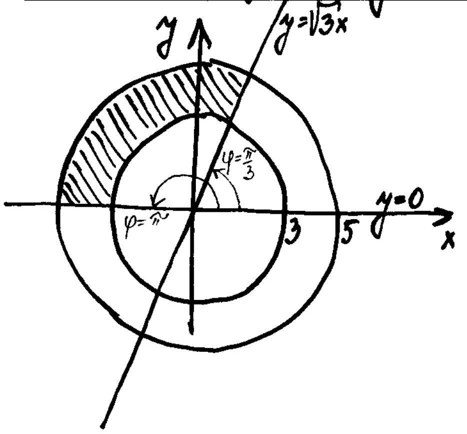
Určete těžiště nehomogenního obrazce M, je-li jeho plošná hustota $\sigma(x,y) = 7x^2y$. M: $9 \le x^2+y^2 \le 25$, $y \ge 0$, $y \ge \sqrt{3}x$.



$9 = x^2+y^2$ je kružnice, $S[0,0]$, $r=\sqrt{9}=3$

$x^2+y^2=25$ je kružnice, $S[0,0]$, $r=\sqrt{25}=5$

$y=0$ je přímka

$y=\sqrt{3}x$ je přímka se směrnicí $\sqrt{3} \Rightarrow \operatorname{tg}\alpha = \sqrt{3}$, $\alpha = 60° = \frac{\pi}{3}$

VZORCE:

$$T(M) = \left[\frac{S_y(M)}{m(M)}, \frac{S_x(M)}{m(M)}\right]$$

$$m(M) = \iint_M \sigma(x,y)\,dx\,dy, \quad S_y(M) = \iint_M x\cdot\sigma(x,y)\,dx\,dy, \quad S_x(M) = \iint_M y\cdot\sigma(x,y)\,dx\,dy$$

Vzhledem k tvaru množiny M zvolíme transformaci do polárních souřadnic: $x = \rho\cos\varphi$

$$y = \rho\sin\varphi \qquad\qquad 3 \le \rho \le 5$$

$$dx\,dy = \underbrace{\rho}_{\text{jakobián transformace}}\,d\rho\,d\varphi \qquad\qquad \frac{\pi}{3} \le \varphi \le \pi$$

$$m(M) = \iint_M \sigma(x,y)\,dx\,dy = \iint_M 7x^2y\,dx\,dy = \int_3^5\left(\int_{\frac{\pi}{3}}^{\pi} 7(\rho\cos\varphi)^2\cdot\rho\sin\varphi\cdot\overset{\text{jakobián}}{\rho}\,d\varphi\right)d\rho =$$

$$= 7\int_3^5 \rho^4\left(\int_{\frac{\pi}{3}}^{\pi}\cos^2\varphi\,\sin\varphi\,d\varphi\right)d\rho = \left|\begin{array}{l}\text{substituce:}\\ \cos\varphi = t\\ -\sin\varphi\,d\varphi = 1\,dt\\ d\varphi = -\frac{1}{\sin\varphi}\,dt\\ \text{přepočítané meze:}\\ \frac{\pi}{3} \xrightarrow{\cos\frac{\pi}{3}} \frac{1}{2}\\ \pi \xrightarrow{\cos\pi} -1\end{array}\right| =$$

$$= 7\cdot\int_3^5\rho^4\left(\int_{\frac{1}{2}}^{-1} t^2\cdot\sin\varphi\cdot\left(-\frac{1}{\sin\varphi}\,dt\right)\right)d\rho = 7\cdot\int_3^5\rho^4\left[-\frac{t^3}{3}\right]_{\frac{1}{2}}^{-1}d\rho =$$

$$= 7\cdot\int_3^5 \varrho^4\cdot\left(-\frac{(-1)^3}{3}-\left(-\frac{\left(\frac{1}{2}\right)^3}{3}\right)\right)d\varrho = 7\cdot\left(\frac{1}{3}+\frac{1}{24}\right)\cdot\int_3^5\varrho^4\,d\varrho =$$

$$= 7\cdot\frac{8+1}{24}\cdot\left[\frac{\varrho^5}{5}\right]_3^5 = \frac{21}{8}\left(\frac{5^5}{5}-\frac{3^5}{5}\right) = \frac{21}{8}\cdot\frac{3125-243}{5} = \frac{30\,261}{20} \doteq \underline{\underline{1513}}$$

- $$S_y(M) = \iint_M x\cdot\sigma(x,y)\,dx\,dy = \iint_M x\cdot 7x^2y\,dx\,dy = \int_3^5\left(\int_{\frac{\pi}{3}}^{\pi} 7(\varrho\cos\varphi)^3\cdot\varrho\sin\varphi\cdot\overset{\text{jakobián}}{\varrho}\,d\varphi\right)d\varrho =$$

$$= 7\int_3^5\varrho^5\left(\int_{\frac{\pi}{3}}^{\pi}\cos^3\varphi\sin\varphi\,d\varphi\right)d\varrho = \left|\begin{array}{l}\text{substituce}\\ \cos\varphi = t\\ \dots\end{array}\right| = 7\cdot\int_3^5\varrho^5\left(\int_{\frac{1}{2}}^{-1}-t^3\,dt\right)d\varrho =$$

$$= 7\cdot\int_3^5\varrho^5\left[-\frac{t^4}{4}\right]_{\frac{1}{2}}^{-1}d\varrho = 7\cdot\left(-\frac{(-1)^4}{4}-\left(-\frac{\left(\frac{1}{2}\right)^4}{4}\right)\right)\cdot\left[\frac{\varrho^6}{6}\right]_3^5 =$$

$$= 7\cdot\left(-\frac{1}{4}+\frac{1}{64}\right)\left(\frac{5^6}{6}-\frac{3^6}{6}\right) = -\frac{32\,585}{8} \doteq \underline{\underline{-4073}}$$

- $$S_x(M) = \iint_M y\,\sigma(x,y)\,dx\,dy = \iint_M y\cdot 7x^2y\,dx\,dy = \int_3^5\left(\int_{\frac{\pi}{3}}^{\pi} 7\cdot(\varrho\cos\varphi)^2(\varrho\sin\varphi)^2\cdot\overset{\text{jakobián}}{\varrho}\,d\varphi\right)d\varrho =$$

$$= 7\cdot\int_3^5\varrho^5\left(\frac{1}{4}\int_{\frac{\pi}{3}}^{\pi}4\cos^2\varphi\cdot\sin^2\varphi\,d\varphi\right)d\varrho \;\underset{\substack{\text{VÍME:}\\ \sin 2\varphi = 2\sin\varphi\cdot\cos\varphi}}{=\!=\!=}\; \frac{7}{4}\int_3^5\varrho^5\left(\int_{\frac{\pi}{3}}^{\pi}\sin^2 2\varphi\,d\varphi\right)d\varrho =$$

VÍME:

$$\sin^2\varphi = \frac{1-\cos 2\varphi}{2}$$

$$\Downarrow$$

$$\sin^2 2\varphi = \frac{1-\cos 4\varphi}{2}$$

$$=\!=\!= \frac{7}{4}\int_3^5\varrho^5\left(\int_{\frac{\pi}{3}}^{\pi}\frac{1-\cos 4\varphi}{2}\,d\varphi\right)d\varrho = \frac{7}{8}\int_3^5\varrho^5\left(\int_{\frac{\pi}{3}}^{\pi}1-\cos 4\varphi\,d\varphi\right)d\varrho =$$

VÍME:

$$\int\cos 4\varphi\,d\varphi = \frac{\sin 4\varphi}{4}$$

$$=\!=\!= \frac{7}{8}\int_3^8\varrho^5\left[\varphi-\frac{\sin 4\varphi}{4}\right]_{\frac{\pi}{3}}^{\pi}d\varrho =$$

$$= \frac{7}{8}\left(\pi - \frac{\sin 4\pi}{4} - \left(\frac{\pi}{3} - \frac{\sin\left(4\cdot\frac{\pi}{3}\right)}{4}\right)\right)\cdot\left[\frac{\varrho^6}{6}\right]_3^5 =$$

$$= \frac{7}{8}\left(\pi - 0 - \frac{\pi}{3} + \frac{-\frac{\sqrt{3}}{2}}{4}\right)\left(\frac{5^6}{6} - \frac{3^6}{6}\right) = -\frac{6517}{72}\left(3\sqrt{3} - 16\pi\right) \doteq \underline{\underline{4079}}$$

$$T(M) \doteq \left[\frac{-4073}{1513} ; \frac{4079}{1513}\right] = \left[-2{,}692 ; 2{,}696\right]$$