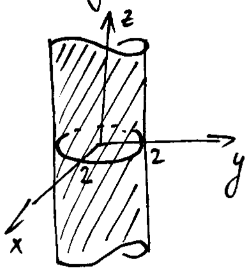
TROJNÝ INTEGRÁL (S TRANSFORMACÍ DO VÁLCOVÝCH SOUŘ.)

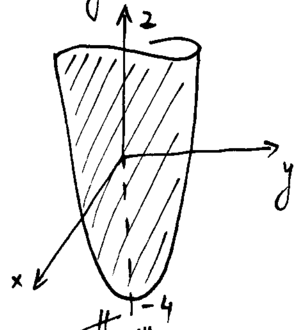
19.4.2017
ÚM FSI VUT

Spočtěte $\iiint\limits_T 3xz\,dx\,dy\,dz$, $T:\ x^2+y^2\le 4,\ x^2+y^2-4\le z\le 5,\ y\le -\sqrt{3}\,x$.

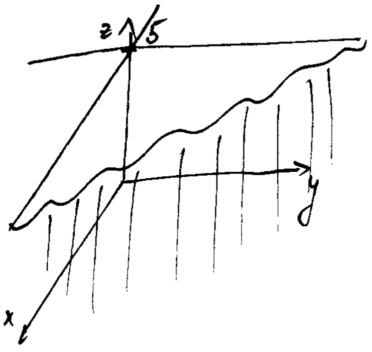
$x^2+y^2\le 4$



$x^2+y^2-4\le z$

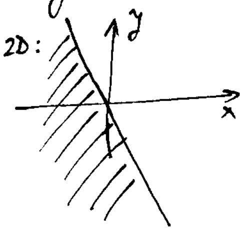


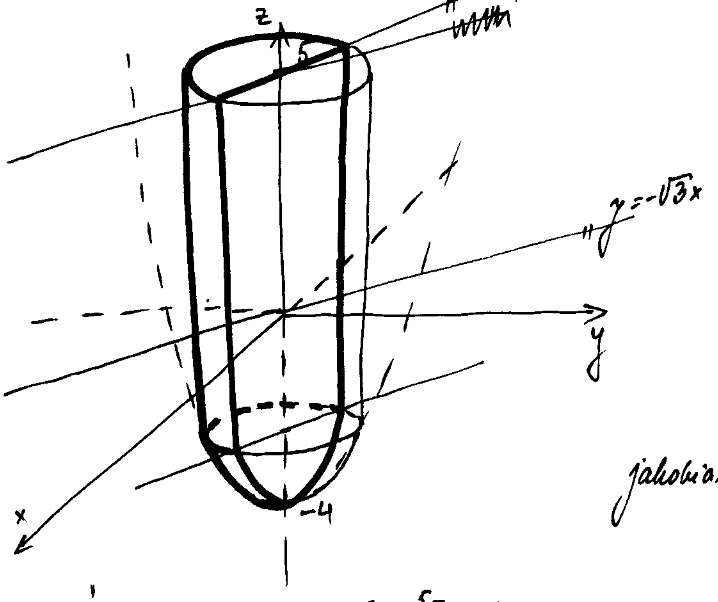
$z\le 5$



$y\le -\sqrt{3}\,x$

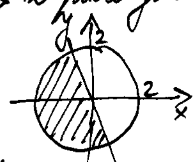
$\operatorname{tg}\alpha = -\sqrt{3}$





Vzhledem k tvaru tělesa T je výhodné zvolit transformaci do válcových souřadnic:

$$x=\rho\cos\varphi \qquad T:\ 0\le\rho\le 2$$
$$y=\rho\sin\varphi \qquad \tfrac{2}{3}\pi\le\varphi\le\tfrac{5}{3}\pi \quad \text{z půdorysu}$$
$$z=w \qquad \underbrace{\rho^2-4}_{\text{paraboloid}}\ \overset{①}{\le}\ w\le \underbrace{5}_{\text{rovina}}$$



jakobián $=\rho$

① $x^2+y^2-4\le z$

$$(\rho\cos\varphi)^2+(\rho\sin\varphi)^2-4\le w$$

$$\iiint\limits_T 3xz\,dx\,dy\,dz=\int_0^2\left(\int_{\frac{2}{3}\pi}^{\frac{5}{3}\pi}\left(\int_{\rho^2-4}^{5} 3\cdot\underbrace{\rho\cos\varphi}_{x}\cdot\underbrace{w}_{z}\cdot\underbrace{\rho}_{\text{jakobián}}\,dw\right)d\varphi\right)d\rho \qquad \rho^2\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1}-4\le w$$

$$\rho^2-4\le w$$

$$=3\int_0^2\rho^2\left(\int_{\frac{2}{3}\pi}^{\frac{5}{3}\pi}\cos\varphi\left(\int_{\rho^2-4}^{5} w\,dw\right)d\varphi\right)d\rho=3\int_0^2\rho^2\left(\int_{\frac{2}{3}\pi}^{\frac{5}{3}\pi}\cos\varphi\left[\frac{w^2}{2}\right]_{\rho^2-4}^{5}d\varphi\right)d\rho=$$

$$=3\cdot\int_0^2\rho^2\left(\frac{5^2}{2}-\frac{(\rho^2-4)^2}{2}\right)\cdot\Big[\sin\varphi\Big]_{\frac{2}{3}\pi}^{\frac{5}{3}\pi}d\rho=\frac{3}{2}\left(\sin\tfrac{5}{3}\pi-\sin\tfrac{2}{3}\pi\right)\cdot\int_0^2\rho^2(25-\rho^4+8\rho^2-16)\,d\rho=$$

$$=\frac{3}{2}\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)\cdot\int_0^2 -\rho^6+8\rho^4+9\rho^2\,d\rho=-\frac{3\sqrt{3}}{2}\left[-\frac{\rho^7}{7}+8\frac{\rho^5}{5}+3\rho^3\right]_0^2=-\frac{3\sqrt{3}}{2}\left(-\frac{2^7}{7}+8\cdot\frac{2^5}{5}+3\cdot 2^3-0\right)=$$

$$=\underline{\underline{-\frac{2988\sqrt{3}}{35}}}\doteq -147{,}9$$