TROJNÝ INTEGRÁL (S TRANSFORMACÍ DO VÁLCOVÝCH SOUŘ.)

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ÚM FSI VUT

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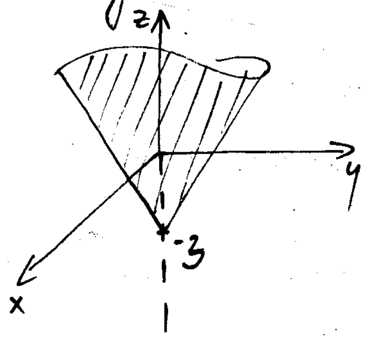
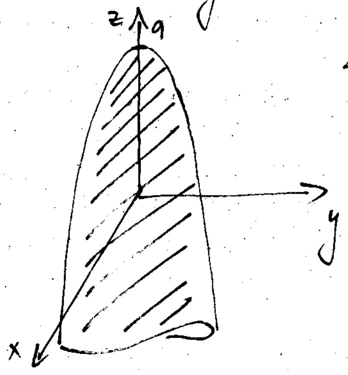
Spočtěte $\iiint\limits_T 4y^2\,dx\,dy\,dz$, $T:\ \sqrt{x^2+y^2}-3 \le z \le 9-x^2-y^2$, $y \ge |x|$.

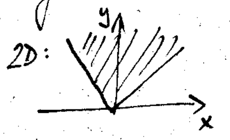
$\sqrt{x^2+y^2}-3 \le z$

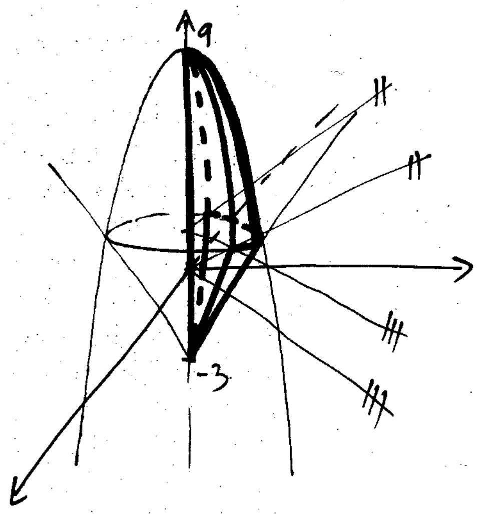


$z \le 9-x^2-y^2$



$y \ge |x|$





Vzhledem k tvaru tělesa T je výhodné zvolit transformaci do válcových souřadnic :

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

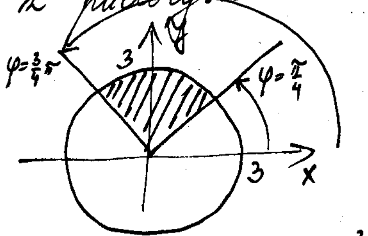
$z = w$

jakobián $= \rho$

T:

$0 \le \rho \le 3$

$\frac{\pi}{4} \le \varphi \le \frac{3}{4}\pi$ } v půdorysu



$\underbrace{\rho-3 \le w}_{\text{kužel}} \overset{①}{} \quad \overset{②}{} \underbrace{w \le 9-\rho^2}_{\text{paraboloid}}$

vypočteme průnik kužele s paraboloidem:

$\underbrace{\sqrt{x^2+y^2}}_{t}-3 = 9\underbrace{-x^2-y^2}_{-t^2}$

$t^2+t-12=0$

$(t-3)(t+4)=0$

$t=3 \qquad \cancel{t=-4}$

poloměr kružnice je 3

① $\sqrt{x^2+y^2}-3 \le z$

$\sqrt{(\rho\cos\varphi)^2+(\rho\sin\varphi)^2}-3 \le w$

$\vdots$

$\rho-3 \le w$

② $z \le 9-x^2-y^2$

$w \le 9-(\rho\cos\varphi)^2-(\rho\sin\varphi)^2$

$\vdots$

$w \le 9-\rho^2$

$$\iiint\limits_T 4y^2\,dx\,dy\,dz = \int_0^3\left(\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\left(\int_{\rho-3}^{9-\rho^2} 4\cdot\underbrace{(\rho\sin\varphi)^2}_{y}\cdot\underset{\text{jakobián}}{\rho}\,dw\right)d\varphi\right)d\rho =$$

$$= 4\int_0^3 \rho^3\left(\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\sin^2\varphi\left(\int_{\rho-3}^{9-\rho^2} 1\,dw\right)d\varphi\right)d\rho =$$

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$$= 4\int_0^3 \rho^3 \left(\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \sin^2\varphi \, \left[w \right]_{\rho-3}^{9-\rho^2} d\varphi \right) d\rho \qquad \underline{\underline{\sin^2\varphi = \frac{1-\cos 2\varphi}{2}}}$$

$$= \cancel{4}^{2} \cdot \int_0^3 \rho^3 \left(9-\rho^2-(\rho-3)\right) \cdot \left(\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \frac{1-\cos 2\varphi}{\cancel{2}} d\varphi \right) d\rho \qquad \underline{\underline{\int \cos 2\varphi \, d\varphi = \frac{\sin 2\varphi}{2}}}$$

$$= 2 \int_0^3 \left(12\rho^3-\rho^5-\rho^4\right) \left[\varphi - \frac{\sin 2\varphi}{2} \right]_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\rho =$$

$$= 2\left(\frac{3}{4}\pi - \frac{\sin\left(2\cdot\frac{3}{4}\pi\right)}{2} - \left(\frac{\pi}{4} - \frac{\sin\left(2\cdot\frac{\pi}{4}\right)}{2} \right) \right) \cdot \left[\cancel{12}^{3} \frac{\rho^4}{\cancel{4}} - \frac{\rho^6}{6} - \frac{\rho^5}{5} \right]_0^3 =$$

$$= 2\left(\frac{\pi}{2} - \frac{-1}{2} + \frac{1}{2} \right) \left(3\cdot 3^4 - \frac{3^6}{6} - \frac{3^5}{5} - 0 \right) =$$

$$= (\pi+2)\, \frac{10\cdot 3^5 - 5\cdot 3^5 - 2\cdot 3^5}{10} = \frac{3^6}{10}(\pi+2) = \underline{\underline{\frac{729(\pi+2)}{10}}} \doteq 374{,}8$$