TROJNÝ INTEGRÁL (S TRANSFORMACÍ DO SFÉRICKÝCH SOUŘADNIC)

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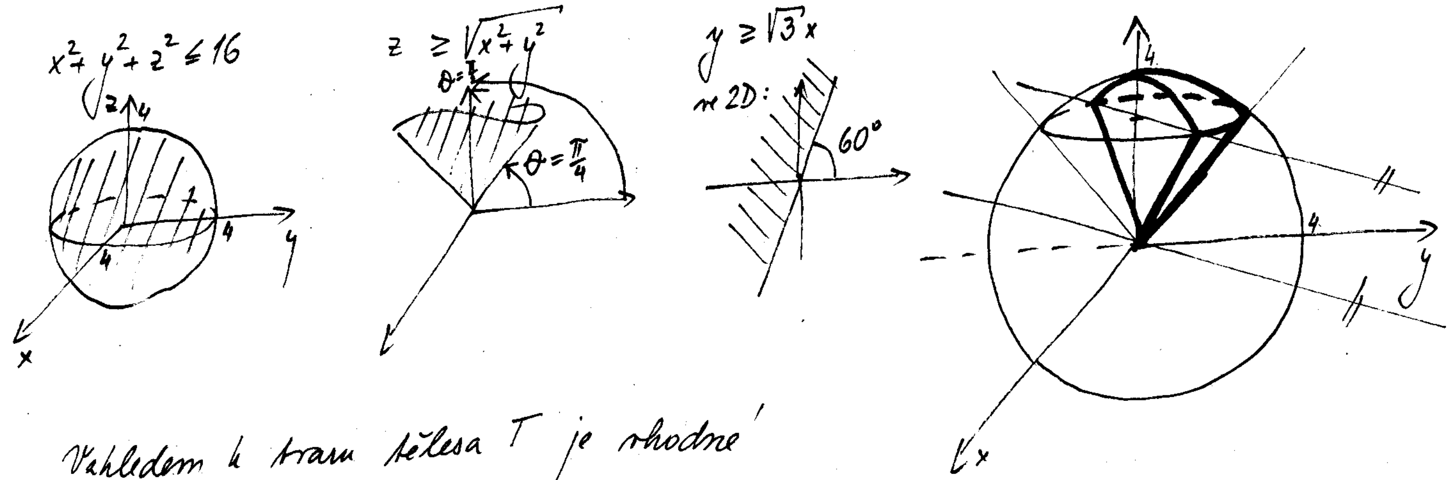
Vypočtěte $\iiint\limits_T 7z\,dx\,dy\,dz$, $\quad T:\ x^2+y^2+z^2 \le 16,\ z \ge \sqrt{x^2+y^2},\ y \ge \sqrt{3}\,x.$



Vzhledem k tvaru tělesa T je vhodné transformovat do sférických souřadnic (pozn. volíme například variantu s odklonem úhlu θ od půdorysny):

$x = \rho\cos\varphi\cos\theta \qquad T:\ 0 \le \rho \le 4 \ \dots$ z prostoru

$y = \rho\sin\varphi\cos\theta \qquad \frac{\pi}{3} \le \varphi \le \frac{4}{3}\pi \ \dots$ z půdorysu

$z = \rho\sin\theta \qquad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$|\text{jakobián}| = \rho^2\cos\theta$

$$\iiint\limits_T 7z\,dx\,dy\,dz = \int_0^4\left(\int_{\frac{\pi}{3}}^{\frac{4}{3}\pi}\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 7\cdot\underbrace{\rho\sin\theta}_{z}\cdot\underbrace{\rho^2\cos\theta}_{|\text{jakobián}|}\,d\theta\right)d\varphi\right)d\rho =$$

$$= \frac{7}{2}\int_0^4\rho^3\left(\int_{\frac{\pi}{3}}^{\frac{4}{3}\pi}\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\underbrace{2\sin\theta\cdot\cos\theta}_{\sin 2\theta}\,d\theta\right)d\varphi\right)d\rho \qquad \underline{\underline{\int\sin 2\theta\,d\theta = \frac{-\cos 2\theta}{2}}}$$

$$= \frac{7}{2}\int_0^4\rho^3\left(\int_{\frac{\pi}{3}}^{\frac{4}{3}\pi}\left[\frac{-\cos 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\varphi\right)d\rho = \frac{7}{2}\left(-\frac{\cos(2\cdot\frac{\pi}{2})}{2}-\left(-\frac{\cos(2\cdot\frac{\pi}{4})}{2}\right)\right)\int_0^4\rho^3\left[\varphi\right]_{\frac{\pi}{3}}^{\frac{4}{3}\pi}d\rho =$$

$$= \frac{7}{4}\left(-(-1)-0\right)\left(\frac{4}{3}\pi-\frac{\pi}{3}\right)\left[\frac{\rho^4}{4}\right]_0^4 = \frac{7}{4}\cdot\pi\left(\frac{4^4}{4}-0\right) = \underline{\underline{112\pi}} \doteq 351{,}9$$