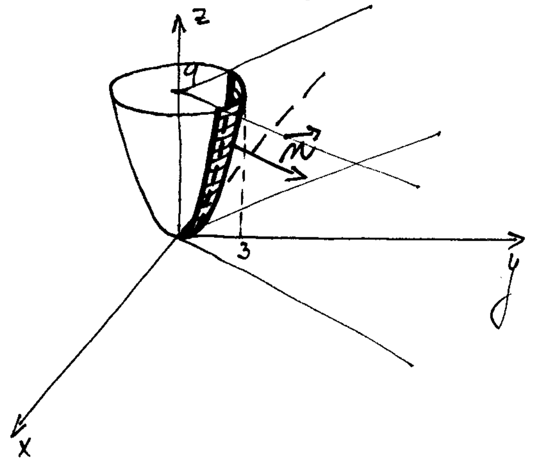
Spočtěte $\iint_{\vec{S}} z\,dy\,dz + x^2y\,dx\,dy$. Plocha $S = \{[x,y,z]\in\mathbb{R}^3 : z = x^2+y^2,\ z\le 9,\ y\ge |x|\}$, normála plochy $S$ míří ven.




Plochou $S$ je část paraboloidu, je dána explicitně rovnicí $z = \underbrace{x^2+y^2}_{f(x,y)}$.

ukazuje v kladném směru osy z $\Uparrow$

Spočteme normálu $\vec{n} = (-f'_x, -f'_y, 1) = (-2x, -2y, 1)$.

Vypočtená normála má nesouhlasnou orientaci se zadanou normálou $\Rightarrow \ominus$

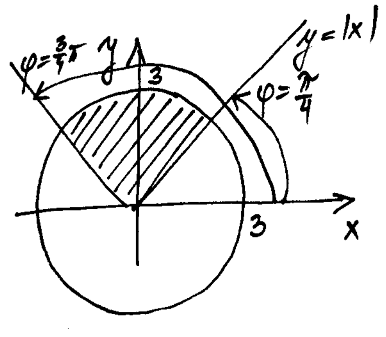
$$\iint_{\vec S} z\,dy\,dz + x^2y\,dx\,dy = \iint_{\vec S} (z,0,x^2y)\,d\vec S = \ominus\iint_{S_{xy}} (\underbrace{x^2+y^2}_{z}, 0, x^2y)\cdot\underbrace{(-2x,-2y,1)}_{\text{normála } \vec n}\,dx\,dy =$$

půdorys plochy $S$

skalární součin

$$= -\iint_{S_{xy}} -2x^3-2xy^2+0+x^2y\,dx\,dy =$$

$S_{xy}$ je část kruhu, transformujeme do polárních souřadnic




$x = \rho\cdot\cos\varphi$

$y = \rho\sin\varphi$

jakobián $= \rho$

$0 \le \rho \le 3$

$\frac{\pi}{4} \le \varphi \le \frac{3}{4}\pi$

$$= -\int_0^3\left(\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\Big(-2(\underbrace{\rho\cos\varphi}_{x})^3 - 2\cdot\underbrace{\rho\cos\varphi}_{x}\cdot(\underbrace{\rho\sin\varphi}_{y})^2 + (\underbrace{\rho\cos\varphi}_{x})^2\underbrace{\rho\sin\varphi}_{y}\Big)\cdot\underbrace{\rho}_{\text{jakobián}}\,d\varphi\right)d\rho =$$

$$= -\int_0^3\rho\left(\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} -2\rho^3\cos^3\varphi - 2\rho^3\sin^2\varphi\cos\varphi + \rho^3\cos^2\varphi\sin\varphi\,d\varphi\right)d\rho =$$

$\cos^3\varphi = \cos^2\varphi\cdot\cos\varphi = (1-\sin^2\varphi)\cdot\cos\varphi$

$$= -\int_0^3 \rho \left( \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} -2\rho^3 (1-\sin^2\varphi)\cos\varphi \, d\varphi - \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} 2\rho^3 \sin^2\varphi \cos\varphi \, d\varphi + \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \rho^3 \cos^2\varphi \sin\varphi \, d\varphi \right) d\rho =$$

$\Downarrow$ subs. $\sin\varphi = t$, $\cos\varphi \, d\varphi = dt$

přepočítáme meze:

$\frac{\pi}{4} \xrightarrow{\sin\frac{\pi}{4}} \frac{\sqrt{2}}{2}$, $\frac{3}{4}\pi \xrightarrow{\sin\frac{3}{4}\pi} \frac{\sqrt{2}}{2}$ $\Rightarrow$ vyjde 0

$\Downarrow$ subs. $\sin\varphi = t$, $\cos\varphi \, d\varphi = dt$

$\frac{\pi}{4} \longrightarrow \frac{\sqrt{2}}{2}$, $\frac{3\pi}{4} \longrightarrow \frac{\sqrt{2}}{2}$ $\Rightarrow$ vyjde 0

$\Downarrow$ subs. $\cos\varphi = t$, $-\sin\varphi \, d\varphi = dt$

$\frac{\pi}{4} \xrightarrow{\cos\frac{\pi}{4}} \frac{\sqrt{2}}{2}$, $\frac{3}{4}\pi \xrightarrow{\cos\frac{3}{4}\pi} -\frac{\sqrt{2}}{2}$

$$= -\int_0^3 \rho \left( 0 - 0 + \rho^3 \int_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} t^2 \cdot (-dt) \right) d\rho = +\int_0^3 \rho^4 \left( \int_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} t^2 \, dt \right) d\rho =$$

$$= \int_0^3 \rho^4 \left[ \frac{t^3}{3} \right]_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} d\rho = \left( \frac{\left(-\frac{\sqrt{2}}{2}\right)^3}{3} - \frac{\left(\frac{\sqrt{2}}{2}\right)^3}{3} \right) \cdot \int_0^3 \rho^4 \, d\rho =$$

$$= \left( -\frac{2\cdot\sqrt{2}}{8\cdot 3} - \frac{2\cdot\sqrt{2}}{8\cdot 3} \right) \cdot \left[ \frac{\rho^5}{5} \right]_0^3 = -\frac{\sqrt{2}}{6} \left( \frac{3^5}{5} - \frac{0^5}{5} \right) = -\frac{\sqrt{2}\cdot 81}{10} \doteq \underline{\underline{-11{,}5}}$$