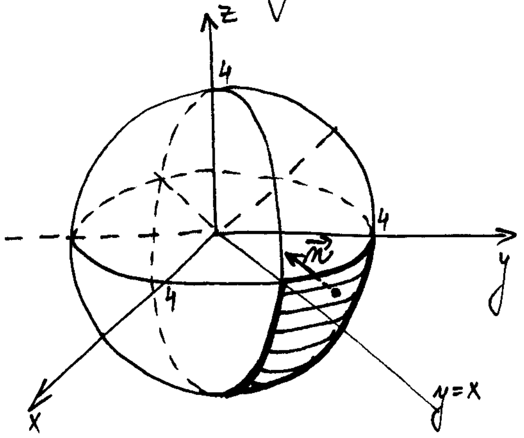
Spočtěte $\iint_{\vec{S}} 1\,dy\,dz + z^2\,dx\,dz + 1\,dx\,dy$, plocha S je část sféry $x^2+y^2+z^2=16$ omezená podmínkami $z \le 0$, $y \ge x$, $x \ge 0$.

Normála plochy S míří dovnitř.



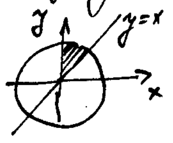
Plochu S by sice bylo možné vyjádřit i explicitně jako $z = -\sqrt{18-x^2-y^2}$, ale u sféry doporučujeme parametrizovat.

Parametrizace plochy (pozn. parametr v volím jako odklon od půdorysny)

$x = 4\cos u \cos v$ $\qquad \frac{\pi}{4} \le u \le \frac{\pi}{2}$

$y = 4\sin u \cos v$

$z = 4\sin v$ $\qquad -\frac{\pi}{2} \le v \le 0$



Vypočteme tečné vektory:

$$\begin{array}{lll} & & \qquad\qquad\qquad\quad \vec{i} \qquad\qquad\qquad \vec{j} \qquad\qquad \vec{k} \end{array}$$

$$\vec{t_u} = (4\cos v\cdot(-\sin u),\ 4\cos v\cdot\cos u,\ 0) = (-4\sin u\cos v,\ 4\cos u\cos v,\ 0)$$

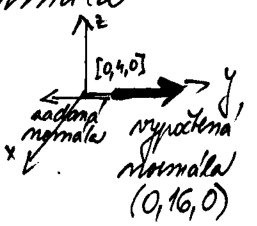
$$\vec{t_v} = (4\cos u(-\sin v),\ 4\sin u(-\sin v),\ 4\cos v) = (-4\cos u\sin v,\ -4\sin u\sin v,\ 4\cos v)$$

Normála $\vec{n} = \vec{t_u} \times \vec{t_v} = (16\cos u\cos^2 v - 0,\ 0 + 16\sin u\cos^2 v,\ 16\sin^2 u\sin v\cos v + 16\cos^2 u\sin v\cdot\cos v) =$

$$= (16\cos u\cos^2 v,\ 16\sin u\cos^2 v,\ 16\sin v\cos v).$$

Abychom byli schopni rozhodnout o vzájemné souhlasnosti / nesouhlasnosti vypočtené normály se zadanou normálou, zvolíme konkrétní vhodné parametry u, v, např. $u = \frac{\pi}{2}$, $v = 0$, které na obrázku odpovídají bodu $[0,4,0]$ na ploše S. V tomto bodě má vypočtená normála tvar

$$\vec{n} \overset{u=\frac{\pi}{2}}{\underset{v=0}{=\!=}} (16\cdot\cos\tfrac{\pi}{2}\cos^2 0,\ 16\cdot\sin\tfrac{\pi}{2}\cos^2 0,\ 16\cdot\sin 0\cdot\cos 0) = (0,16,0) \Rightarrow$$



Vypočtená normála má nesouhlasnou orientaci vzhledem k zadané normále $\Rightarrow$ $(-)$

PLOŠNÝ INTEGRÁL 2. DRUHU (ŘEŠENÝ POMOCÍ PARAMETRIZACE PLOCHY)

$$\iint_{\vec{\varphi}} 1\,dy\,dz + z^2\,dx\,dz + 1\,dx\,dy = \iint_{\vec{\varphi}} (1, z^2, 1)\,\vec{dS} =$$

$$= \ominus \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\int_{-\frac{\pi}{2}}^{0} (1, \underbrace{16\sin^2 v}_{z^2}, 1) \cdot \underbrace{(16\cos u \cos^2 v, 16\sin u \cos^2 v, 16 \sin v \cos v)}_{\text{normála}}\, dv \right) du =$$

skalární součin

$$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\int_{-\frac{\pi}{2}}^{0} 16\cos u \cos^2 v + 16^2 \sin u \sin^2 v \cos^2 v + 16 \sin v \cos v \; dv \right) du =$$

$$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(16\cos u \int_{-\frac{\pi}{2}}^{0} \cos^2 v\, dv + 16^2 \sin u \int_{-\frac{\pi}{2}}^{0} \sin^2 v \cos^2 v\, dv + 8 \cdot \int_{-\frac{\pi}{2}}^{0} 2 \sin v \cos v\, dv \right) du =$$

$$\sin 2v = 2\sin v \cos v$$

$$\sin^2 v = \frac{1-\cos 2v}{2}$$

$$\cos^2 v = \frac{1+\cos 2v}{2}$$

$$- \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\cancel{16}^{\,8} \cos u \int_{-\frac{\pi}{2}}^{0} \frac{1+\cos 2v}{\cancel{2}}\, dv + \cancel{16^2}^{\,64} \sin u \int_{-\frac{\pi}{2}}^{0} \underbrace{\frac{1-\cos 2v}{\cancel{2}} \cdot \frac{1+\cos 2v}{\cancel{2}}}_{\substack{1-\cos^2 2v = \\ = 1 - \frac{1+\cos 4v}{2} = \frac{1}{2} - \frac{\cos 4v}{2}}}\, dv + 8 \cdot \int_{-\frac{\pi}{2}}^{0} \sin 2v\, dv \right) du =$$

$$\int \sin 2v\, dv = \frac{-\cos 2v}{2}$$

$$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(8\cos u \int_{-\frac{\pi}{2}}^{0} 1 + \cos 2v\, dv + \cancel{64}^{\,32} \sin u \int_{-\frac{\pi}{2}}^{0} \frac{1}{\cancel{2}} - \frac{\cos 4v}{\cancel{2}}\, dv + 8 \left[-\frac{\cos 2v}{2} \right]_{-\frac{\pi}{2}}^{0} \right) du =$$

$$\int \cos 2v\, dv = \frac{\sin 2v}{2}$$

$$\int \cos 4v\, dv = \frac{\sin 4v}{4}$$

$$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(8\cos u \left[v + \frac{\sin 2v}{2} \right]_{-\frac{\pi}{2}}^{0} + 32 \sin u \left[v - \frac{\sin 4v}{4} \right]_{-\frac{\pi}{2}}^{0} + 8 \left(-\frac{\cos(2\cdot 0)}{2} - \left(-\frac{\cos\left(2\cdot\left(-\frac{\pi}{2}\right)\right)}{2} \right) \right) \right) du =$$

$$= -\left(8 \cdot \left(0 + \frac{\sin(2\cdot 0)}{2} - \left(-\frac{\pi}{2} + \frac{\sin\left(2\cdot\left(-\frac{\pi}{2}\right)\right)}{2} \right) \right) \cdot \left[\sin u \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + 32 \left(0 - \frac{\sin(4\cdot 0)}{4} - \left(-\frac{\pi}{2} - \frac{\sin\left(4\cdot\left(-\frac{\pi}{2}\right)\right)}{4} \right) \right) \left[-\cos u \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \right.$$

$$\left. + 8 \cdot \left(-\frac{1}{2} + \frac{(-1)}{2} \right) \right) = -\left(8 \cdot \frac{\pi}{2} \left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4} \right) + 32 \cdot \frac{\pi}{2} \left(-\cos\frac{\pi}{2} - \left(-\cos\frac{\pi}{4} \right) \right) + 8 \cdot (-1) \right) =$$

$$= -4\pi \left(1 - \frac{\sqrt{2}}{2} \right) - 16\pi \frac{\sqrt{2}}{2} + 8 \doteq \underline{\underline{-31{,}2}}$$