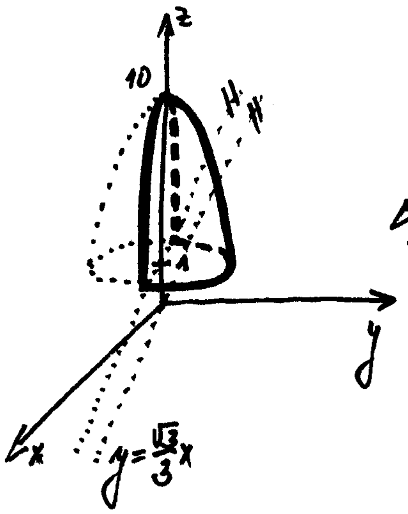
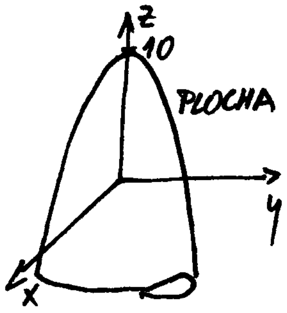
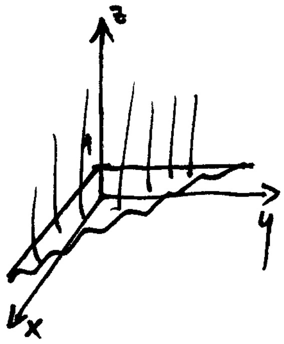
PLOŠNÝ INTEGRÁL 1. DRUHU – OBSAH PLOCHY DANÉ EXPLICITNĚ

Plošným integrálem 1. druhu spočtěte obsah plochy $\mathcal{S}$, kde
$\mathcal{S} = \{[x,y,z] \in \mathbb{R}^3 : \; z = 10 - x^2 - y^2 \,, \; z \geq 1 \,, \; y \geq \frac{\sqrt{3}}{3}x \}$.

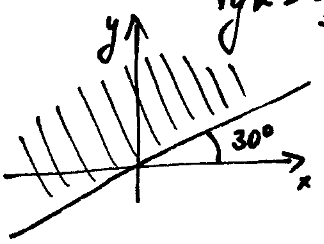






ve 2D: $y \geq \frac{\sqrt{3}}{3}x$

$\operatorname{tg}\alpha = \frac{\sqrt{3}}{3}$



Obsah plochy $S(\mathcal{S}) = \iint\limits_{\mathcal{S}} 1 \, dS$

Plocha $\mathcal{S}$ je dána explicitně jako $z = 10 - x^2 - y^2$

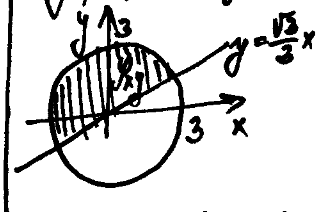
Nachystáme si normálu:

$\vec{n} = \left(-(-2x), -(-2y), 1\right) = (2x, 2y, 1)$

$|\vec{n}| = \sqrt{(2x)^2 + (2y)^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$

$$S(\mathcal{S}) = \iint\limits_{\mathcal{S}} 1 \, dS \overset{\text{DEFINICE}}{=} \iint\limits_{\mathcal{S}_{xy}} 1 \cdot \underbrace{\sqrt{4x^2+4y^2+1}}_{|\vec{n}|} \, dx\,dy = \left| \begin{array}{l} \mathcal{S}_{xy} \text{ je půdorys plochy } \mathcal{S}, \text{ tedy} \\ \text{Volíme transformaci do} \\ \text{polárních souřadnic:} \end{array} \right| =$$

$\mathcal{S}_{xy}$ je půdorys plochy $\mathcal{S}$, tedy



Volíme transformaci do polárních souřadnic:

$x = \rho\cos\varphi$ $\qquad 0 \leq \rho \leq 3$

$y = \rho\sin\varphi$

jakobián $= \rho$ $\qquad \frac{\pi}{6} \leq \varphi \leq \frac{7}{6}\pi$

poloměr: $10 - x^2 - y^2 = 1$

$9 = x^2 + y^2$

$$= \int_0^3 \left(\int_{\frac{\pi}{6}}^{\frac{7}{6}\pi} \sqrt{4\cdot(\underbrace{\rho\cos\varphi}_{x})^2 + 4(\underbrace{\rho\sin\varphi}_{y})^2 + 1} \cdot \underbrace{\rho}_{\text{jakobián}} \, d\varphi \right) d\rho =$$

$$= \int_0^3 \rho \left(\int_{\frac{\pi}{6}}^{\frac{7}{6}\pi} \sqrt{4\rho^2(\underbrace{\cos^2\varphi + \sin^2\varphi}_{1}) + 1} \, d\varphi \right) d\rho =$$

$$= \int_0^3 \rho\cdot\sqrt{4\rho^2+1} \left(\int_{\frac{\pi}{6}}^{\frac{7}{6}\pi} 1 \, d\varphi \right) d\rho =$$

$$= \left[\varphi\right]_{\frac{\pi}{6}}^{\frac{7}{6}\pi} \cdot \int_0^3 \sqrt{4\rho^2+1}\cdot\rho\, d\rho = \left| \begin{array}{l} \text{substituce} \\ 4\rho^2+1 = t^2 \\ 8\rho\, d\rho = 2t\, dt \\ \rho\, d\rho = \frac{t}{4} dt \\ 0 \xrightarrow{4\cdot 0^2+1} t^2 = 1 \Rightarrow t = 1 \\ 3 \xrightarrow{4\cdot 3^2+1} t^2 = 37 \Rightarrow t = \sqrt{37} \end{array} \right| =$$

$$= \left(\frac{7}{6}\pi - \frac{\pi}{6}\right) \int_1^{\sqrt{37}} t \cdot \frac{t}{4} dt = \pi \cdot \frac{1}{4} \left[\frac{t^3}{3}\right]_1^{\sqrt{37}} =$$

$$= \frac{\pi}{4}\left(\frac{(\sqrt{37})^3}{3} - \frac{1^3}{3} \right) = \underline{\underline{\frac{\pi}{12}\left(37\cdot\sqrt{37} - 1\right)}} \doteq 58{,}7$$