PLOŠNÝ INT. 1. DRUHU – PLOCHA JE DANÁ EXPLICITNĚ

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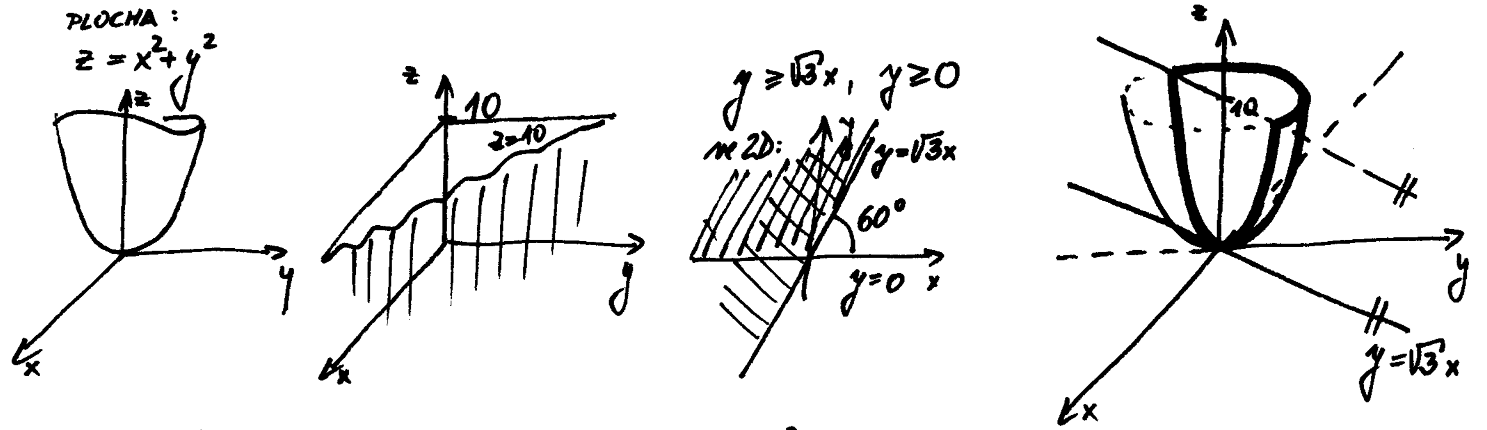
UM FSI VUT

Spočtěte $\iint_{\varphi} xy\,dS$, kde $\varphi = \{[x,y,z] \in \mathbb{R}^2 : z = x^2+y^2,\ z \le 10,\ y \ge \sqrt{3}x,\ y \ge 0\}$



Plocha φ je dána explicitně jako $z = \underbrace{x^2+y^2}_{g(x,y)}$.

Nachystáme si normálu:

$\vec{n} = (-g'_x, -g'_y, 1) = (-2x, -2y, 1)$

$|\vec{n}| = \sqrt{(-2x)^2 + (-2y)^2 + 1^2} = \sqrt{4x^2+4y^2+1}$

$$\iint_{\varphi} xy\,dS \overset{DEF.}{=} \iint_{\varphi_{xy}} xy \cdot \underbrace{\sqrt{4x^2+4y^2+1}}_{|\vec{n}|}\,dx\,dy =$$

vzhledem k tvaru φ_{xy} (to je půdorys plochy φ) je vhodné transformovat do polárních souřadnic:

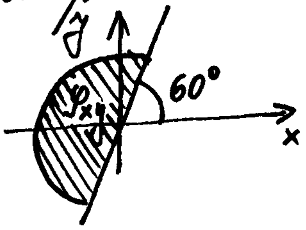
$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

jakobián $= \rho$

$0 \le \rho \le \sqrt{10}$

$\frac{\pi}{3} \le \varphi \le \pi$



poloměr:

$x^2+y^2 = 10$

$\Downarrow$

$r = \sqrt{10}$

$$= \int_0^{\sqrt{10}} \left(\int_{\frac{\pi}{3}}^{\pi} \left(\rho\cos\varphi\ \rho\sin\varphi \cdot \sqrt{4(\rho\cos\varphi)^2 + 4(\rho\sin\varphi)^2 + 1} \right) \cdot \rho\,d\varphi \right) d\rho = \int_0^{\sqrt{10}} \rho^3 \left(\int_{\frac{\pi}{3}}^{\pi} \cos\varphi\sin\varphi\sqrt{4\rho^2+1}\,d\varphi \right) d\rho =$$

jakobián

$$= \frac{1}{2}\int_0^{\sqrt{10}} \rho^3\sqrt{4\rho^2+1} \left(\int_{\frac{\pi}{3}}^{\pi} 2\cos\varphi\sin\varphi\,d\varphi \right) d\rho = \frac{1}{2}\int_0^{\sqrt{10}} \rho^3\sqrt{4\rho^2+1} \left(\int_{\frac{\pi}{3}}^{\pi} \sin 2\varphi\,d\varphi \right) d\rho =$$

$\sin 2\varphi = 2\sin\varphi\cos\varphi$

$\int \sin 2\varphi\,d\varphi = \frac{-\cos 2\varphi}{2}$

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$$= \frac{1}{2} \int_0^{\sqrt{10}} \rho^3 \sqrt{4\rho^2+1} \cdot \left[-\frac{\cos 2\varphi}{2} \right]_{\frac{\pi}{3}}^{\pi} d\rho =$$

$$= \frac{1}{2} \left(-\frac{\cos(2\cdot\pi)}{2} - \left(-\frac{\cos(2\cdot\frac{\pi}{3})}{2} \right) \right) \cdot \int_0^{\sqrt{10}} \rho^3 \sqrt{4\rho^2+1}\, d\rho =$$

$$= \frac{1}{2} \left(-\frac{1}{2} + \frac{-\frac{1}{2}}{2} \right) \cdot \int_1^{\sqrt{41}} \frac{t^2-1}{4} \cdot \sqrt{t^2} \cdot \frac{t}{4} dt =$$

$$= \frac{1}{2} \cdot \left(-\frac{3}{4} \right) \cdot \frac{1}{16} \int_1^{\sqrt{41}} (t^2-1) \cdot t \cdot t\, dt =$$

$$= -\frac{3}{128} \int_1^{\sqrt{41}} t^4 - t^2\, dt = -\frac{3}{128} \left[\frac{t^5}{5} - \frac{t^3}{3} \right]_1^{\sqrt{41}} =$$

$$= -\frac{3}{128} \left(\frac{(\sqrt{41})^5}{5} - \frac{(\sqrt{41})^3}{3} - \left(\frac{1^5}{5} - \frac{1^3}{3} \right) \right) =$$

$$= -\frac{3}{128} \left(\frac{41^2 \cdot \sqrt{41} \cdot 3 - 41 \cdot \sqrt{41} \cdot 5 - 3 + 5}{15} \right) =$$

$$= -\frac{1}{640} \left(2 + 5043\sqrt{41} - 205\sqrt{41} \right) =$$

$$= -\frac{1}{320} \left(1 + 2419\sqrt{41} \right) \doteq -48{,}4$$

$$\rho^2 = \frac{t^2-1}{4}$$

substituce

$$4\rho^2+1 = t^2$$

$$8\rho\, d\rho = 2t\, dt$$

$$\rho\, d\rho = \frac{t}{4} dt$$

$$0 \xrightarrow{4\cdot 0^2+1} t^2 = 1 \Rightarrow t = 1$$

$$\sqrt{10} \xrightarrow{4(\sqrt{10})^2+1} t^2 = 41 \Rightarrow t = \sqrt{41}$$