Cartesian product

Given two sets A, B we define their Cartesian product $A \times B$

$$\boxed{A \times B = \left\{ (a,b) \mid a \in A \wedge b \in B \right\}}$$

$A \times B$ is the set of all the pairs whose first element is in A and second in B. Note that (a,b) and (b,a) are different pairs.

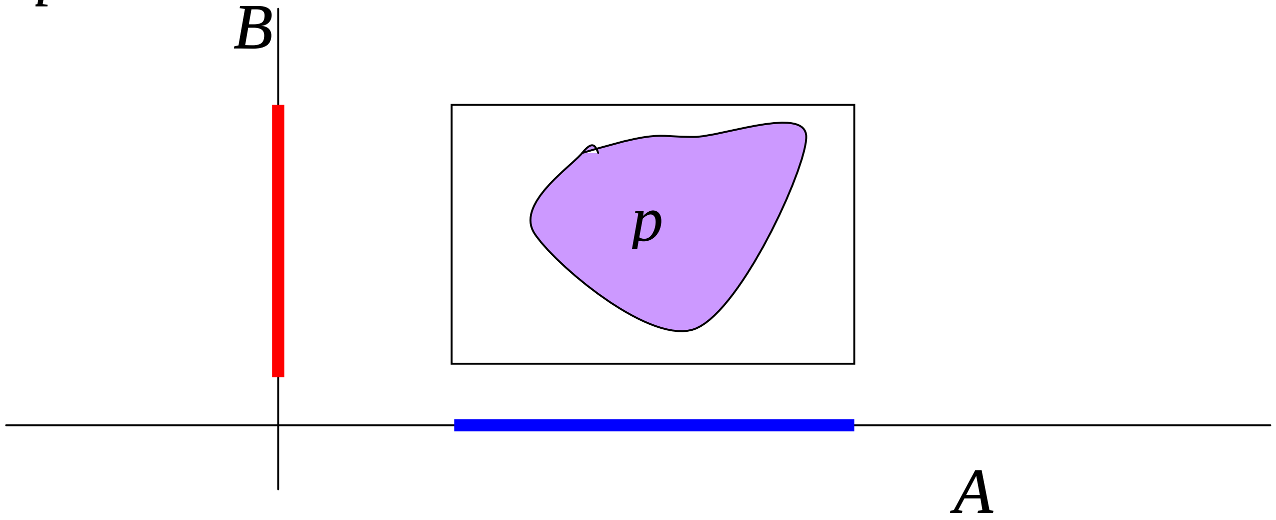
Correspondence

For two sets A, B a correspondence π between A and B is a subset of the Cartesian product of A and B, formally

$$p \subseteq A \times B$$

We say that two elements $a \in A, b \in B$ are in correspondence π

if $(a, b) \in p$. We also write $a \, \pi \, b$.



Examples of correspondences

P is the set of all the products a company manufactures.

M is the set of all the materials the company uses for manufacturing

We define product p and material m to be in correspondence π if material m is used by the company to manufacture product p.

Let E be the set of all chemical elements.

We define the correspondence ρ between materials M and chemical elements E: a material m is in correspondence with an element e if m contains e.

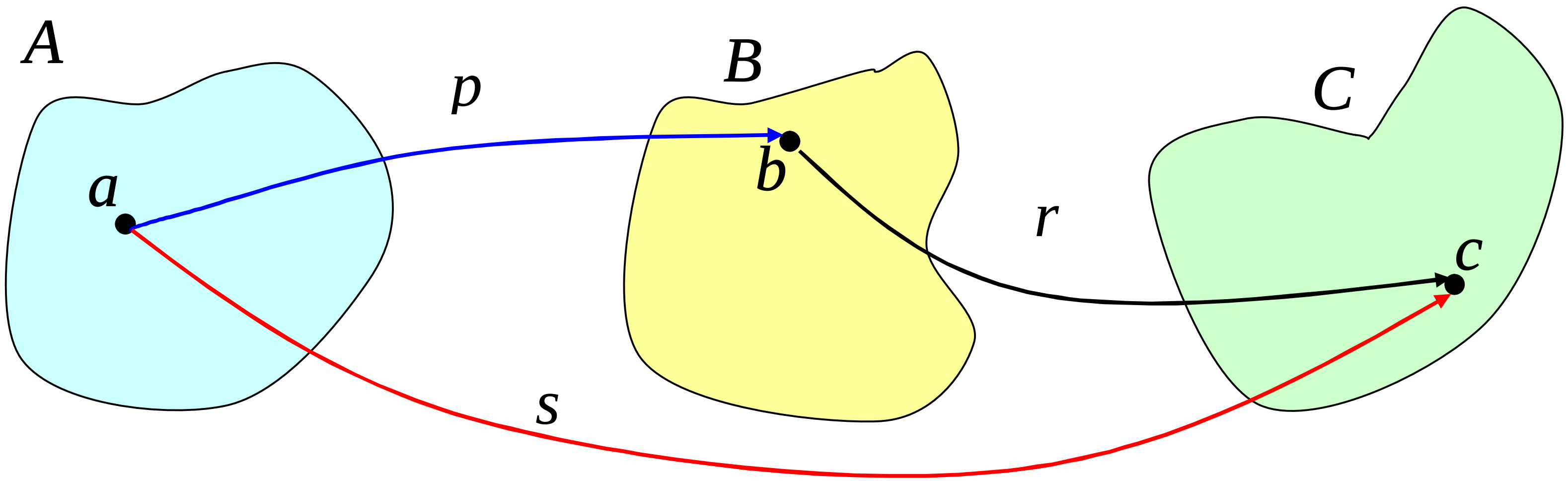
Composition of correspondences

π –correspondence between A and B

ρ – correspondence between B and C.

$$s = \{(a,c) \in A \times C \mid \exists b : ((a,b) \in p \wedge (b,c) \in r)\}$$

$s = r \circ p$ is a correspondence beween A and C



Example

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Product *p* and material *m* are in correspondence π if material *m* is used by the company to manufacture product *p*.

E is the set of all chemical elements.

Material *m* is in correspondence ρ with element *e* if *m* contains *e*.

The composition $r \circ p$ may be described as the set of all the pairs (*p*,*e*) of materials and elements such that the product *p* contains the element *e*.

Relation

Correspondence ρ between two identical copies of a set *A* is called a relation on *A*.

$$r \subseteq A \times A$$

Example

Let Z be a set of all integers, that is,

$$Z = \{\ldots,-4,-3,-2,-1,0,1,2,3,4,\ldots\}$$

then the relation of divisibility $|$ is defined as:

$(a,b) \in |$ or $a \mid b$ if a divides b.

Relations with special properties

A relation $r \subseteq A \times A$ is called

reflexive, if $\forall a \in A : (a,a) \in r$

transitive, if $\forall a,b,c \in A : ((a,b) \in r \wedge (b,c) \in r) \Rightarrow (a,c) \in r$

symmetric, if $\forall a,b \in A : (a,b) \Rightarrow (b,a)$

antisymmetric, if $\forall a,b \in A : ((a,b) \in r \wedge (b,a) \in r) \Rightarrow a = b$

Example - order

Let Z be the set of all integers and let $a \le b$ be the relation "a is less than or equal to b" defined in the usual way. Then $\le$ is reflexive, transitive, and antisymmetric.

Indeed, for any z_1, z_2, z_3, we have

$$z_1 \le z_1$$

$$z_1 \le z_2 \wedge z_2 \le z_3 \Rightarrow z_1 \le z_3$$

$$z_1 \ge z_2 \wedge z_2 \le z_1 \Rightarrow z_1 = z_2$$

A relation on A that is reflexive, transitive, and antisymmetric is called a *partial order* on A usually denoted $\le$. If, moreover, we have $a \le b$ for any a,b in A, $\le$ is called an *order* on A.

Example - equivalence

Let A be a set and a system $P_1, P_2, \ldots, P_n$ of subsets of A be given such that $\bigcup_{i=1}^{n} P_i = A$ and $i \neq j \Rightarrow P_i \cap P_j = \emptyset$ (we say that the sets $P_1, P_2, \ldots, P_n$ are pair-wise disjunct). Such a system of subsets is called a ***partition*** of A.

Define a relation ε on A as follows: $(a,b) \in e \Leftrightarrow \exists i : a,b \in P_i$

It is easy to prove that ε is reflexive, transitive, and symmetric

Any relation ε on A that is reflexive, transitive, and symmetric is called an ***equivalence*** on A

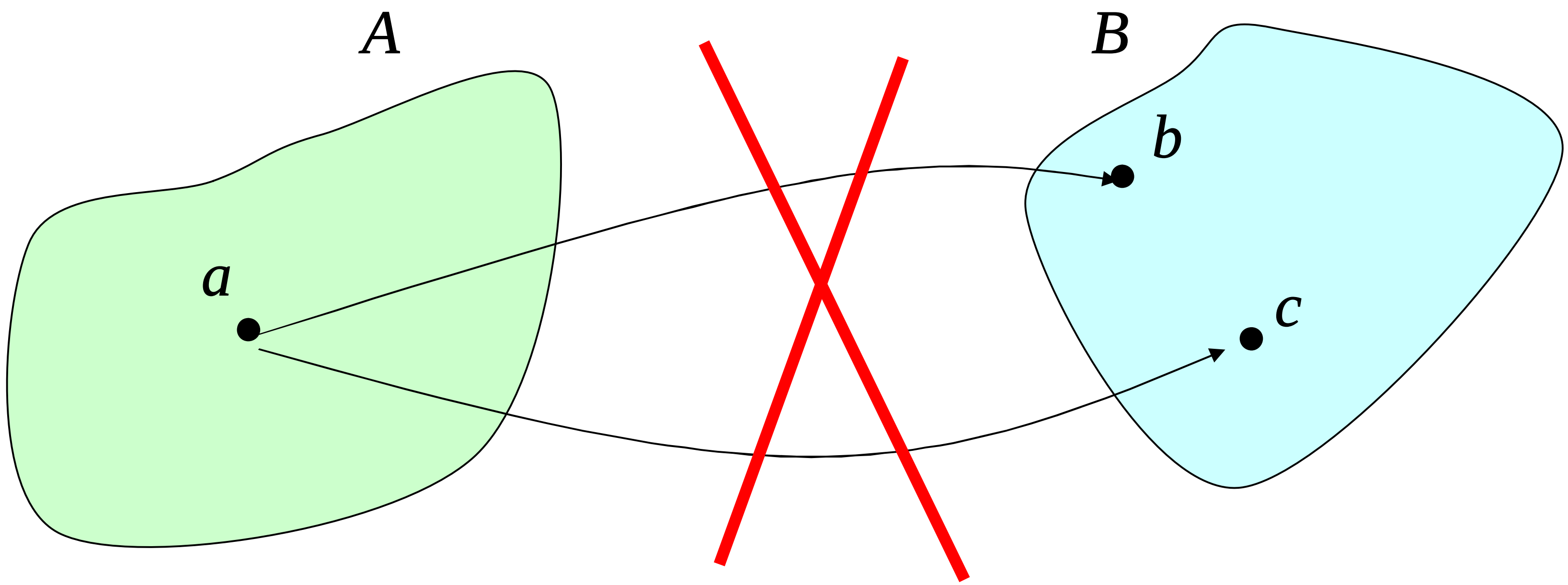
Mapping

Let *A*, *B* be sets and φ a correspondence between *A* and *B*.
If φ has the following properties:

$$\forall a \in A, \forall b, c \in B : ((a,b) \in j \wedge (a,c) \in j\ ) \Rightarrow b = c$$

$$\forall a \in A : \exists b \in B : (a,b) \in j$$

we say that φ is a mapping of *A* into *B*



Special notation for mappings:

Instead of $\varphi \subseteq A \times B$ we write $\boxed{\varphi : A \to B}$

Instead of $(a,b) \in \varphi$ or $a\,\varphi\,b$ we write $\boxed{b = \varphi(a)}$

We also say that φ takes or sends a to b.

Domain, range, image:

If $\varphi : A \to B$ is a mapping, then the set A is called the ***domain*** of φ.

The set R(φ) defined as $R(\varphi) = \{ y \in B \mid \exists x \in A : \varphi(x) = y \}$

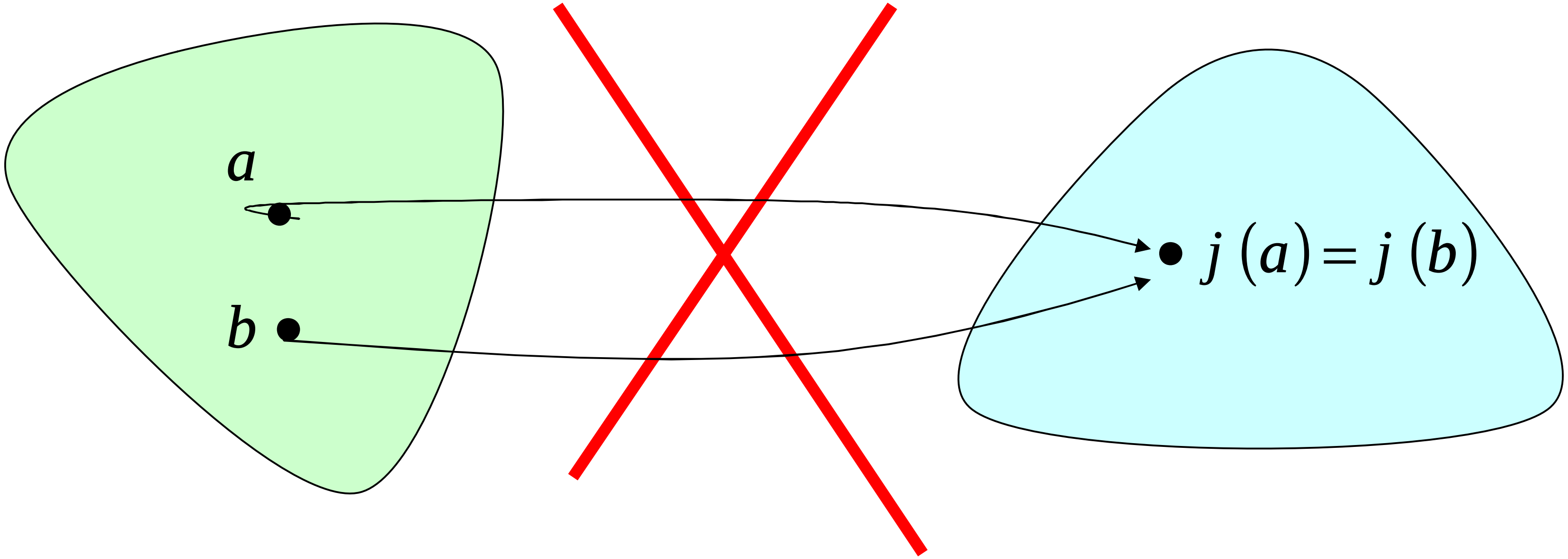
is called the ***range*** of φ.

If $b = \varphi(a)$, then we say that b is the ***image*** of a

One-to-one mapping (injection)

A mapping $j : A \;\mathbf{a}\; B$ is called *one-to-one* or an *injection* if

$$j(a) = j(b) \Rightarrow a = b$$



"Onto" mapping (surjection)

We say that a mapping

$$\varphi : A \to B$$

maps A onto (rather than into) B if $R(\varphi) = B$

Such a mapping is sometimes called ***surjection***.

Bijection, cardinality

A one-to-one mapping φ of *A* onto *B* is sometimes called a ***bijection***

If, for two sets *A*, *B*, a bijection $j : A \ \mathbf{a} \ B$ exists, we say that *A* and *B* have the same cardinality

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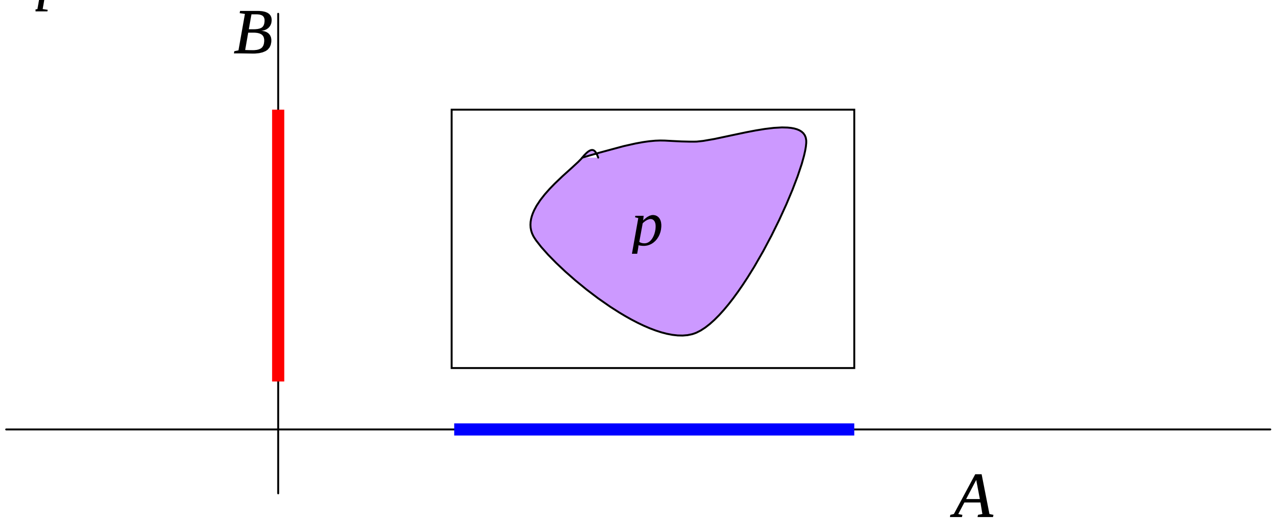

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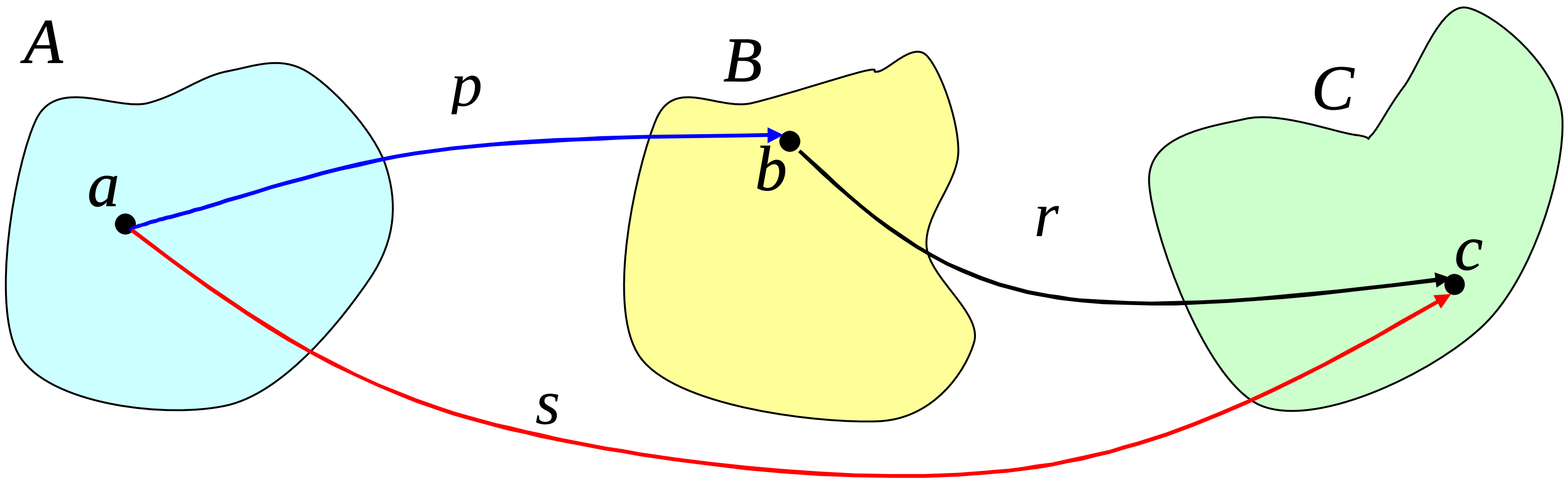

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Let *A*, *B* be sets and φ a correspondence between *A* and *B*.
If φ has the following properties:

$$\forall a \in A, \forall b, c \in B : \big((a,b) \in j \wedge (a,c) \in j\big) \Rightarrow b = c$$

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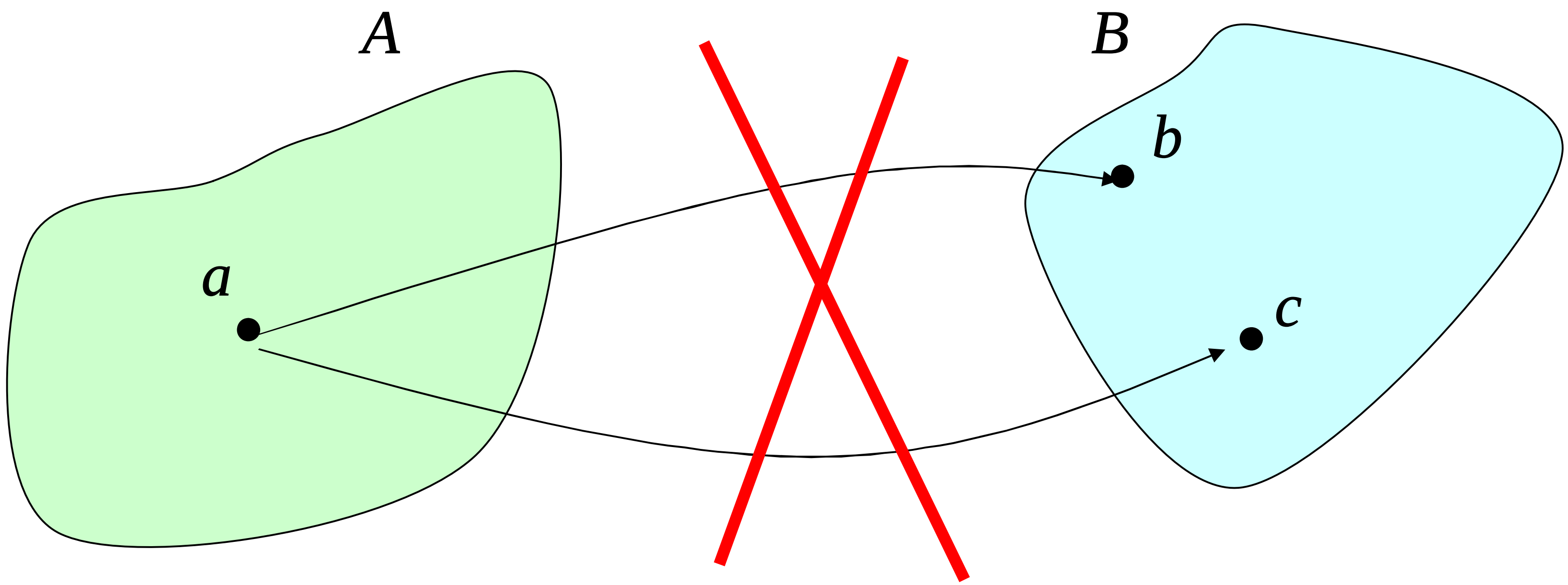

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Instead of $(a,b)\in j$ or $a\; j\; b$ we write $\boxed{b = j\,(a)}$

We also say that φ takes or sends *a* to *b*.

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If $\varphi : A \to B$ is a mapping, then the set A is called the ***domain*** of φ.

The set R(φ) defined as $R(\varphi) = \{ y \in B \mid \exists x \in A : \varphi(x) = y \}$

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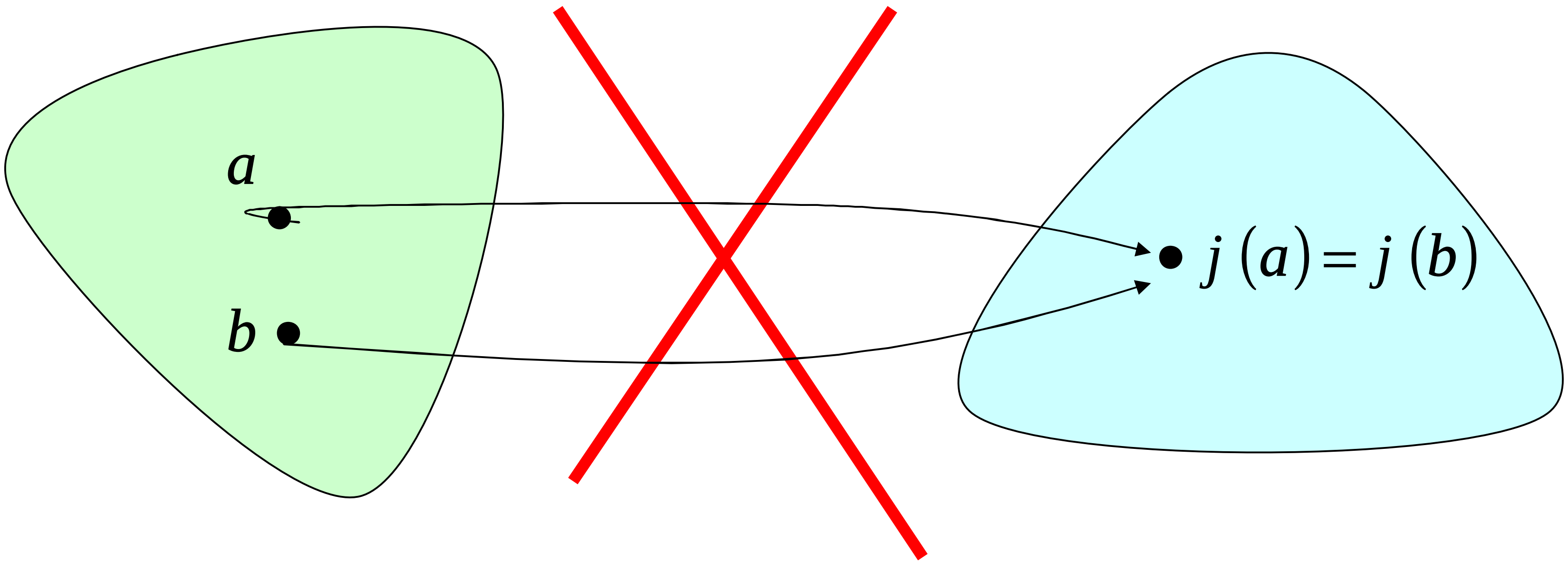

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