Infinite sets

We say that a set A is infinite if a proper subset B exists of A such that there is a bijection

$$\varphi : A \mapsto B$$

It is easy to see that no set with a finite number of elements can satisfy such a condition whereas, for example, for the set A={1,2,3,...} we can define the a set B={2,3,4,...} and a mapping φ as follows: $\varphi(i) = \varphi(i+1),\ i = 1,2,\ldots$ This mapping is a bijection.

<u>Integers</u>

The infinite set $\boldsymbol{N} = \{1,2,3,...\}$ is called *natural numbers* or *positive integers*.

The equation $a + x = b$ has no solution for some positive integers such as $3 + x = 1$ and so we add 0 and negative integers to obtain the set $Z = \{\mathbf{K}, -3, -2, -1, 0, 1, 2, 3, \mathbf{K}\}$ of integers.

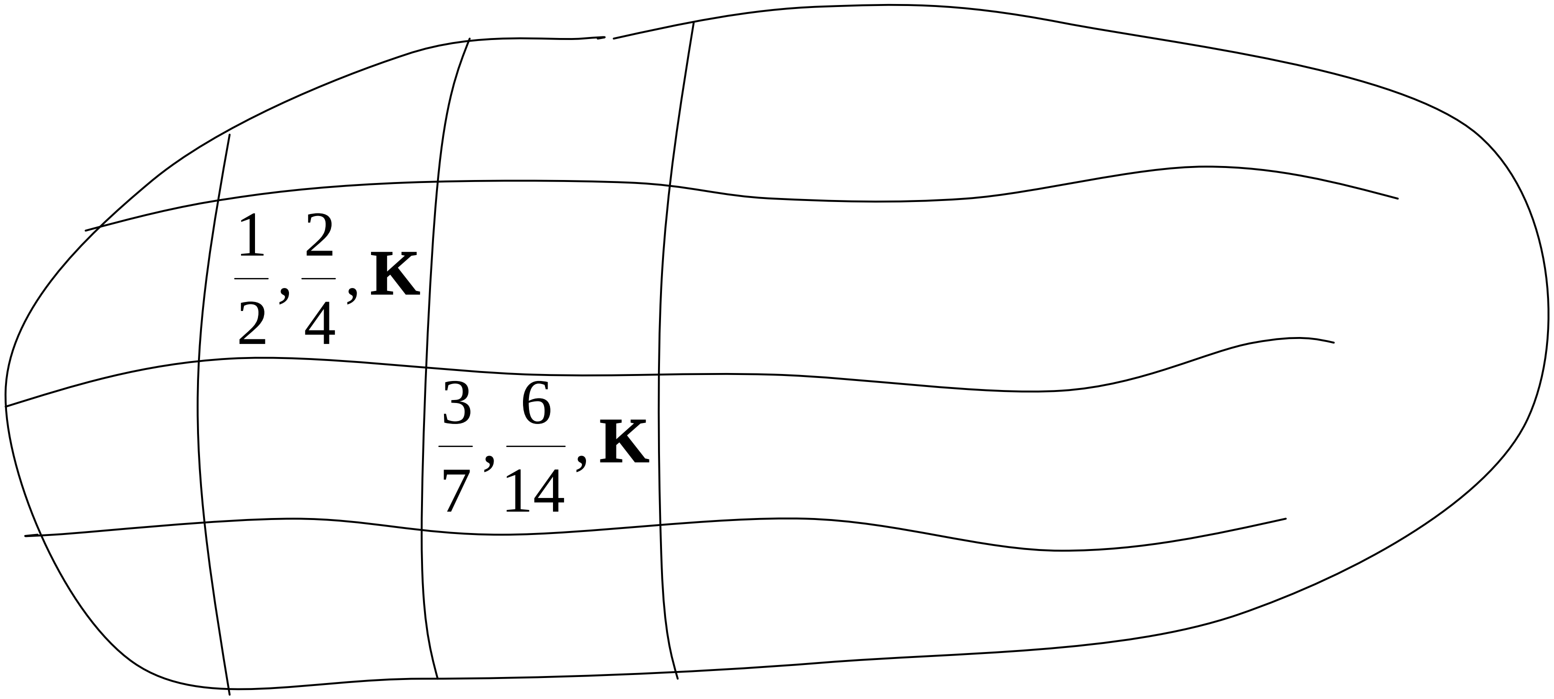
The set Z is ordered by the relation $\leq$. If, for $a, b \in \mathbf{Z}$, we have $a \leq b$, we say that *a* is less than or equal to *b*. We also write $b \geq a$. and say that *b* is greater than or equal to *a*.

Note that, while $\boldsymbol{N}$ has the least element 1, in $\boldsymbol{Z}$**,** no such element exists.

Rational numbers

For some integers, such as $b = 3$ and $a = 7$, the solution of the equation $b.x = a$ is not an integer so we proceed as follows.

Let us consider the set S of all the pairs a/b of integers



We shall consider every two such pairs *a*/*b* and *c*/*d* identical writing $a/b \approx c/d$ if $a.d = c.b$. It is easy to prove that ≈ is an equivalence and as such defines a partition *Q* of the set of all pairs *a*/*b*. The set *Q* is then referred to as rational numbers.

For $q = \left\{ \frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots \right\}$ we can find, say, $\frac{a_0}{b_0}$ such that

a_0 and b_0, $b_0 > 0$, are relatively prime integers, that is, their only common divisor is 1.

For rational numbers $q_1 = \frac{a}{b}, q_2 = \frac{c}{d}, \quad b, d > 0$ we put

$$\boxed{q_1 \le q_2 \quad \Leftrightarrow \quad ad \le cd}$$

This correctly defines an order on the set of all rational numbers

Instead of rational numbers as sets we rather work with the „fractions“ representing them. Further we define the usual arithmetic operations:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+cb}{bd} \qquad b,d\neq 0$$

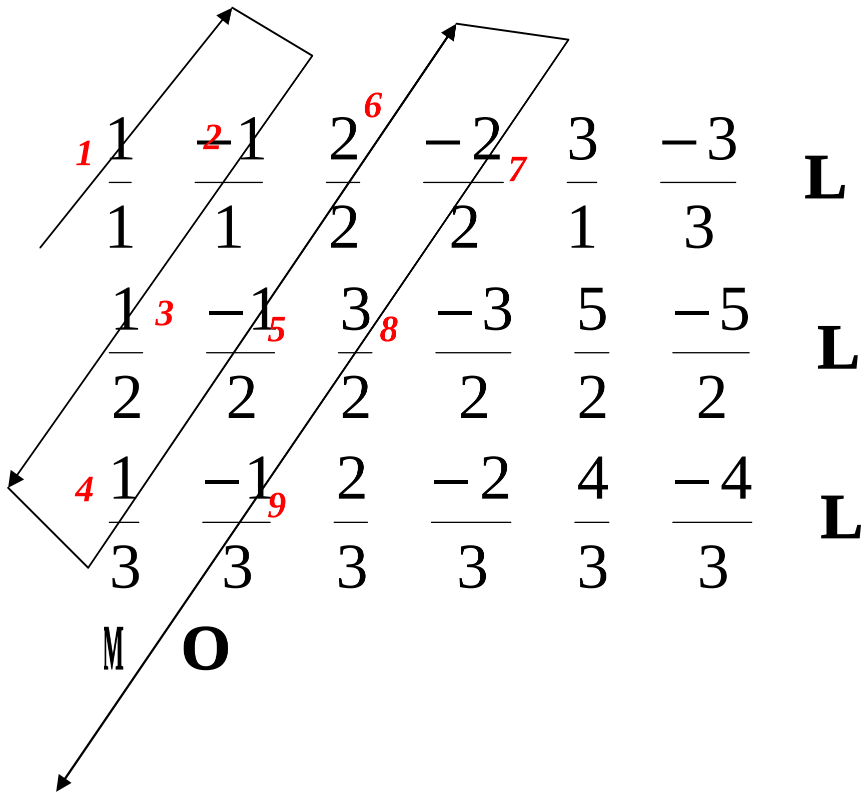
$$\frac{a}{b}-\frac{c}{d}=\frac{ad-cb}{bd} \qquad b,d\neq 0$$

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd} \qquad b,d\neq 0$$

$$\frac{a}{b}:\frac{c}{d}=\frac{ad}{bc} \qquad b,c,d\neq 0$$

Cardinality of rational numbers

The set ***Q*** of rational numbers has the same cardinality as the set ***N*** of natural numbers.



For every two different rational numbers $a/b \leq c/d$, there is a rational number p/q such that $p/q \neq a/b, c/d$ and $a/\mathrm{b} \leq p/q \leq c/d$.

Proof.

We have $a.d \leq b.c$, a/b ^{1} c/d, which we will denote by $a.d < b.c$.

Let us consider the rational number $p/q = (a.d + b.c)/(2.b.d)$.

$$a.q = a.2.b.d = 2.b.a.d < b.a.d + b.b.c < b.(a.d + b.c) = b.p$$

In a similar way, we can prove that $p.d < c.q$.

On account of the preceding property, we say that the set $\boldsymbol{Q}$ of rational numbers is dense.

Although the rational numbers are densely arranged, there are still "holes" between them as can be demonstrated by the following argument:

$\sqrt{2}$ is not a rational number.

Suppose $\sqrt{2} = \frac{a}{b}$ where $b > 0, a, b$ are relatively prime.

$\sqrt{2} = \frac{a}{b} \Rightarrow 2b^2 = a^2 \Rightarrow a = 2k$ for some integer k. The last implication follows from the fact that the square of an odd number is again odd.

Then $2b^2 = (2k)^2 = 4k^2 \Rightarrow b^2 = 2k^2$ and $b = 2m$ for some integer m.

Thus we get a contradiction since a and b are relatively prime.

Real numbers

By "filling up the holes" in the set of *rational numbers*, we can construct a set of real numbers denoted by ***R***.

This process is too sophisticated to fit in the scope of the present course and so we will just settle for listing the properties of ***R*.**

By the *e-neighbourhood* of a real number x we understand the interval

$$(x-e, x+e).$$

Properties of real numbers

- R is ordered by the relation $\leq$

- Every equation $a + x = b$ where a,b are real has a real solution

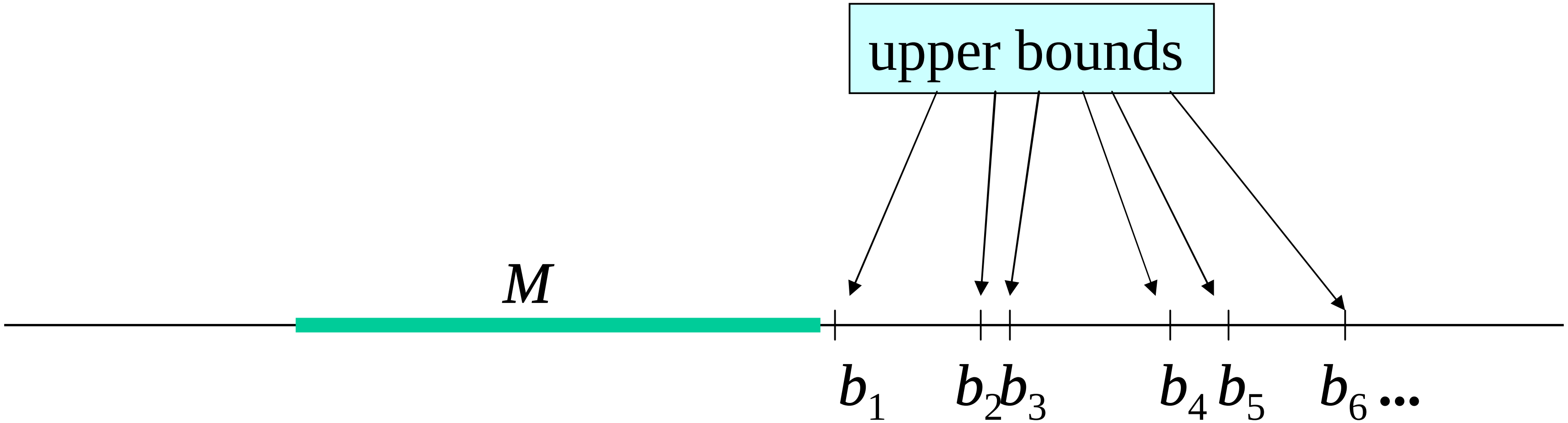
- Every equation $a \,.\, x = b$ where a,b are real and $a \neq 0$, has a real solution

- Every equation $x^2 = b$ where $b > 0$ is real has a real solution

- In every ε-neighbourhood of a real number a, there are an infinite number of rational numbers and an infinite number of real numbers that are not rational.

Least upper bound

Let M be a set of real numbers. We say that b is an *upper bound* of M if $b \geq x$ for every $x \in M$.

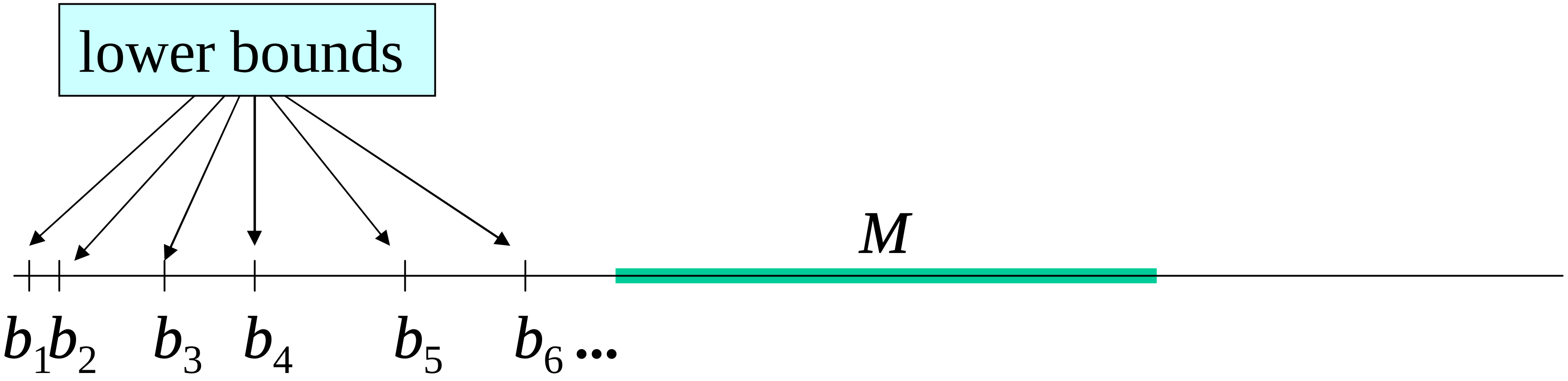


Let us denote by $U(M)$ the set of all the upper bounds of M. It is another property of real numbers that every such set has the least element l, that is, $l \in U(M)$, $\quad l \leq x,\ x \in U(M)$

Such an l is called the ***least upper bound*** of LUB or sometimes a ***supremum***.

Greatest lower bound

Let M be a set of real numbers. We say that b is an *upper bound* of M if $b \geq x$ for every $x \in M$.



Let us denote by $L(M)$ the set of all the lower bounds of M. It is another property of real numbers that each such set has the greatest element h, that is, $h \in U(M)$, $h \geq x$, $x \in U(M)$

Such an h is called the ***greatest lower bound*** of GLB or sometimes an ***infimum***.

Cardinality of ***R***

The set ***R*** of real numbers has a greater cardinality that the set of natural numbers ***N***.

Every real number r in the interval (0,1) can be written as

$r = 0.x_1x_2x_3...$ where x_1, x_2, x_3, ... are digits 0 to 9.

$r_1 = 0.458796280$ ...

$r_2 = 0.221087755$ ... $r = 0.5382$ ...

$r_3 = 0.997214120$ ...

$r_4 = 0.521136987$ ...

r cannot be any of the numbers r_1, r_2, r_3, ...

...

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