System of *m* linear equations with *n* real unknownss

coefficients

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n-1}x_{n-1} + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots a_{2n-1}x_{n-1} + a_{2n}x_n = b_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn-1}x_{n-1} + a_{mn}x_n = b_m$$

right-hand sides

unknowns

Example for *n* = 3, *m* = 3

$$
\begin{aligned}
3x - y \;\; + 5z &= 16 \\
-x - 2y + z \;\; &= 7 \\
x + \;\; y + z \;\; &= 0
\end{aligned}
$$

substituting method $\Rightarrow$ solution $x = 1, y = -3, z = 2$

For larger systems, the substituting method is highly impractical.

We shall introduce a more sophisticated procedure called the Gauss elimination method.

Basic observations

Each system S of m equations defines a set E of solutions

Set E of solutions remains the same if

- any two equations in S are swapped
- any equation in S is multiplied by a non-zero number
- any equation in S is added to any other in S

Some systems may be more easy to solve than others

Example of an easy system

$$
\begin{aligned}
-6x_1 + 5x_2 - x_3 + 8x_4 &= -5 \\
2x_2 + x_3 + 2x_4 &= 4 \\
x_3 + 3x_4 &= 1 \\
2x_4 &= -1
\end{aligned}
$$

m by *n* matrix

element

rows

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

columns

Examples

a 3 by 3 real matrix

$$\begin{pmatrix} 3 & -4 & 6 \\ 2 & 0 & 3.14 \\ -2.152 & p & 5 \end{pmatrix}$$

a 2 by 4 complex matrix

$$\begin{pmatrix} -5 & 3+2i & 1.78-1.43i & 5i \\ 2-3.12i & 7 & -1-i & 23 \end{pmatrix}$$

Multiplying a row of a matrix by a real number

$$\begin{array}{cc} & \\ c \longrightarrow & \\ & \\ & \\ & \end{array}\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ ca_{21} & ca_{22} & ca_{23} & \cdots & ca_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

Adding a row of a matrix to another row

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{11}+a_{31} & a_{12}+a_{32} & a_{13}+a_{33} & \cdots & a_{1n}+a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

Staircase matrix

$$\begin{pmatrix} 3 & -4 & 0 & 1 & 3 \\ 0 & 2 & 5 & -5 & 1 \\ 0 & 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & 0 & 10 \end{pmatrix}$$

The transpose of a matrix

n columns

We say that A^T is the transpose of A

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

m rows

m columns

the rows and columns change places

$$a_{ij} = a^T_{ji}$$

$$A^T = \begin{pmatrix} a^T_{11} & a^T_{12} & a^T_{13} & \cdots & a^T_{1m} \\ a^T_{21} & a^T_{22} & a^T_{23} & \cdots & a^T_{2m} \\ a^T_{31} & a^T_{32} & a^T_{33} & \cdots & a^T_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a^T_{n1} & a^T_{n2} & a^T_{n3} & \cdots & a^T_{nm} \end{pmatrix}$$

n rows

Multiplying a matrix by a scalar

If every element of a matrix A has been multiplied by the same scalar c (a real number), we say that A has been multiplied by the scalar c.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

$$cA = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & ca_{23} & \cdots & ca_{2n} \\ ca_{31} & ca_{32} & ca_{33} & \cdots & ca_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & ca_{m3} & \cdots & ca_{mn} \end{pmatrix}$$

Adding two matrices

if A is an m by n matrix and B also an m by n matrix we can define their sum

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{pmatrix}$$

$$A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} & \cdots & a_{2n}+b_{2n} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+a_{33} & \cdots & a_{3n}+b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & a_{m3}+b_{m3} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

Multiplying two matrices

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1p} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2p} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mp} \end{pmatrix} \qquad B=\begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & b_{p3} & \cdots & b_{pn} \end{pmatrix}$$

$$C=\begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & c_{m3} & \cdots & c_{mn} \end{pmatrix}$$

$$c_{ij} = \sum_{k=1}^{p} a_{ip} b_{pj}$$

$$A \approx (m \times p), \quad B \approx (p \times n), \quad C \approx (m, n)$$

System matrix, extended system matrix

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \cdots a_{1n-1}x_{n-1} + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots a_{2n-1}x_{n-1} + a_{2n}x_n = b_2 \\
\quad \vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn-1}x_{n-1} + a_{nn}x_n = b_n
\end{array}
$$

$$(A) \qquad\qquad (A \mid \boldsymbol{b})$$

$$
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{pmatrix}
\qquad
\left(\begin{array}{ccccc|c}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\
a_{31} & a_{32} & a_{33} & \cdots & a_{31} & b_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m
\end{array}\right)
$$

system matrix

extended system matrix

Gauss elimination procedure

assuming $m < n$

- establish the extended system matrix
- use the first row to enforce zeroes in the first column under the first element in the first column
- use the resulting second row to enforce zeroes in the second column under the second element in the second column
- proceed in a similar way for columns 3, 4, ..., m
- in a "smooth case" the result should be a staircase matrix where finding the solution should be an easy matter

Example

three equations with three unknowns - "smooth" case

$$\begin{aligned} 3x - y \quad + 5z &= 16 \\ -x - 2y + z \quad &= 7 \\ x + \; y + z \quad &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 5 & 16 \\ -1 & -2 & 1 & 7 \\ 1 & 1 & 1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 7 \\ 0 & 0 & -6 & -12 \end{array}\right)$$

a single solution $x = 1, y = -3, z = 2$

Example (a "tricky" case)

three equations in three variables - no solution

$$\begin{aligned} 3x - 2y &+ 4z = -1 \\ -x + y &+ z = 0 \\ 5x - 4y &+ 2z = 1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 3 & -2 & 4 & -1 \\ -1 & 1 & 1 & 0 \\ 5 & -4 & 2 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

no *x*, *y*, and *z* can be found such that $0x + 0y + 0z = -2$

Example (a tricky case)

4 equations with 4 unknowns - an infinite number of solutions

$$\begin{aligned} 4x_1 + 8x_2 + 4x_3 + 4x_4 &= 8 \\ 3x_1 + 2x_2 + 7x_3 - 5x_4 &= -6 \\ 3x_1 + 6x_2 + 3x_3 + 3x_4 &= 6 \\ 5x_1 + 7x_2 + 8x_3 - x_4 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 &= -3s + 3t - 2 \\ x_2 &= s - 2t - 3 \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

$$\left[\begin{array}{cccc|c} 4 & 8 & 4 & 4 & 8 \\ 3 & 2 & 7 & -5 & -6 \\ 3 & 6 & 3 & 3 & 6 \\ 5 & 7 & 8 & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 4 & 8 & 4 & 4 & 8 \\ 0 & -4 & 4 & -8 & -12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

In order to identify the smooth and various types of tricky cases, the notion of the rank of a matrix must be introduced.

First we introduce the notion of linear dependence and independence of rows (columns) of a matrix):

$$\begin{array}{l} \boldsymbol{a}_1 \rightarrow \\ \boldsymbol{a}_2 \rightarrow \\ \boldsymbol{a}_3 \rightarrow \\ \\ \boldsymbol{a}_m \rightarrow \end{array} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{pmatrix}$$

$$\vec{o} \rightarrow (0,0,0,\ldots,0)$$

$$c\boldsymbol{a}_{i_1} = (ca_{i1}, ca_{i2}, ca_{i3}, \ldots, ca_{in})$$

We say that rows $\boldsymbol{a}_{i_1}, \boldsymbol{a}_{i_2}, \boldsymbol{a}_{i_3}, \ldots, \boldsymbol{a}_{i_k}$ are linearly independent if

$$(*) \qquad c_1\boldsymbol{a}_{i_1} + c_2\boldsymbol{a}_{i_2} + c_3\boldsymbol{a}_{i_3} + \ldots + c_k\boldsymbol{a}_{i_k} = \vec{o}$$

can only be satisfied if

$$c_1 = c_2 = c_3 = \cdots = c_k = 0$$

otherwise if, say, $c_1 \neq 0$ we can write

$$\boldsymbol{a}_{i_1} = \frac{c_2}{c_1}\boldsymbol{a}_{i_2} + \frac{c_3}{c_1}\boldsymbol{a}_{i_3} + \ldots + \frac{c_k}{c_1}\boldsymbol{a}_{i_n} = \boldsymbol{o}$$

and say that $\overrightarrow{a_{i_1}}$ is a linear combination of $\overrightarrow{a_{i_2}}, \overrightarrow{a_{i_3}}, \ldots, \overrightarrow{a_{i_k}}$

The rank of a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

If in *A*, there exist *k* independent rows and any *k* + 1 or more rows are not independent, we say that the rank *r* (*A*) of A is *k*.

A way of determining the rank of a matrix is to transform the matrix into a staircase matrix and see how many non-zero rows (rows containing at least one non-zero element) it contains.

$$A := \begin{bmatrix} 4 & 8 & 4 & 4 & 8 \\ 3 & 2 & 7 & -5 & -6 \\ 3 & 6 & 3 & 3 & 6 \\ 5 & 7 & 8 & -1 & 1 \end{bmatrix} \Longrightarrow B := \begin{bmatrix} 4 & 8 & 4 & 4 & 8 \\ 0 & -4 & 4 & -8 & -12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2 nonzero rows

the rank of A is 2

A matrix A has the same rank as its transpose A^T

$$r\left(A^T\right) = r(A)$$

$$A := \begin{bmatrix} 4 & 8 & 4 & 4 & 8 \\ 3 & 2 & 7 & -5 & -6 \\ 3 & 6 & 3 & 3 & 6 \\ 5 & 7 & 8 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 8 & 4 & 4 & 8 \\ 0 & -4 & 4 & -8 & -12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2 rows

$$A^T = \begin{bmatrix} 4 & 3 & 3 & 5 \\ 8 & 2 & 6 & 7 \\ 4 & 7 & 3 & 8 \\ 4 & -5 & 3 & -1 \\ 8 & -6 & 6 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 3 & 3 & 5 \\ 0 & -4 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2 rows $\neq 0$

The Frobenius theorem

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n-1}x_{n-1} + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n-1}x_{n-1} + a_{2n}x_n &= b_2 \qquad (*) \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn-1}x_{n-1} + a_{mn}x_n &= b_m \end{aligned}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \quad A|\vec{b} = \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{31} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

The system (*) has a solution if and only if $r(A) = r\left(A|\vec{b}\right)$

How many solutions there are to a system of linear equations?

- if $r(A) \neq r(A|\vec{b})$, that is, $r(A) < r(A|\vec{r})$, the system has no solution

- if the system has a solution, then if $r(A) = n$ where n is the number of unknowns, the system has exactly one solution, otherwise it has an infinite number of solutions. These solutions can then be established by introducing some of the variables as parameters.

Note that the system matrix cannot have a rank greater than the number of its variables. Why?