

Solving the following system of 2 equations with two unknowns

$$a_{11}x + a_{12}y = b_1$$

$$a_{21}x + a_{22}y = b_2$$

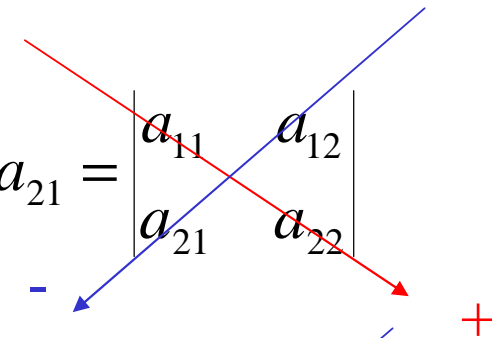
by the Gauss elimination method, we obtain

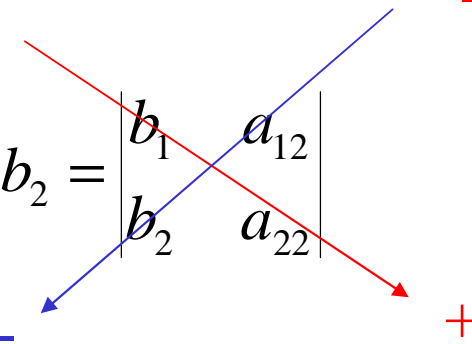
$$x = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}}$$

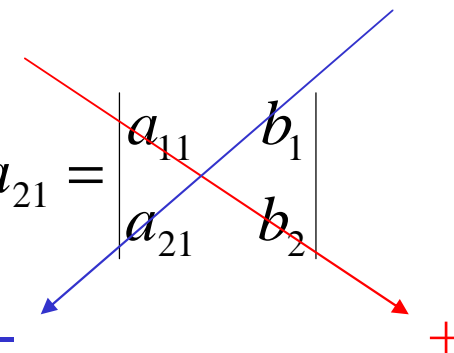
$$y = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}$$

provided that  $a_{11} a_{22} - a_{12} a_{21} \neq 0$

The expressions in the numerators and denominators can be conveniently expressed by the following schemes:

$$a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$


$$b_1a_{22} - a_{12}b_2 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$


$$a_{11}b_2 - b_1a_{21} = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$


The expressions

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_1 \end{vmatrix}$$

are referred to as the determinants of the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad A_1 = \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix} \qquad A_2 = \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_1 \end{pmatrix}$$

sometimes written as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |A_1| = \det(A_1) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{etc.}$$

Thus, for the system

$$a_{11}x + a_{12}y = b_1$$

$$a_{21}x + a_{22}y = b_2$$

we can write

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

provided that

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

Similarly, for a system of three equations with three unknowns,

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

we could easily establish by the Gauss elimination method that,  
for example,

$$x = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

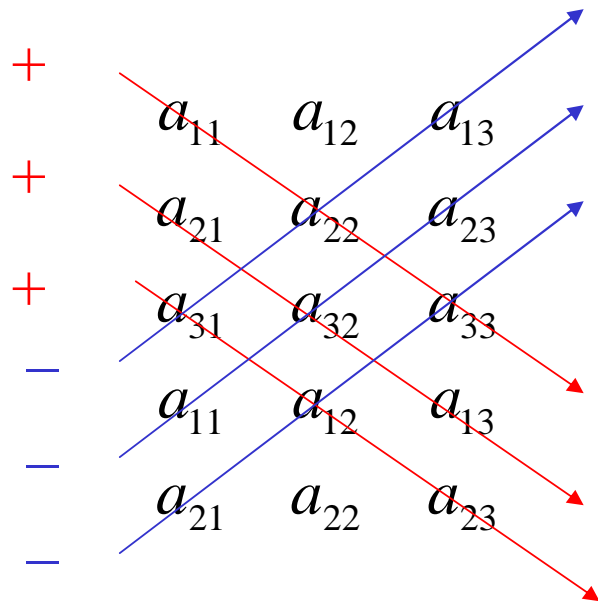
provided that the  
denominator is not  
equal to zero

Where

$$\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = b_1 a_{22} a_{33} + b_2 a_{32} a_{13} + b_3 a_{12} a_{23} - \\ - (b_3 a_{22} a_{13} + b_1 a_{32} a_{23} + b_2 a_{12} a_{33})$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} a_{22} a_{33} + a_{21} a_{32} a_{13} + a_{31} a_{12} a_{23} - \\ - (a_{31} a_{22} a_{13} + a_{11} a_{32} a_{23} + a_{21} a_{12} a_{33})$$

Sometimes we use the following auxiliary scheme



The expression

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - \\ - (a_{31}a_{22}a_{13} + a_{11}a_{32}a_{23} + a_{21}a_{12}a_{33})$$

is called the determinant of a 3 by 3 matrix and, for

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{denoted by}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \det(A) = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$



Similar schemes can also be devised for calculating systems of linear algebraic equations of orders higher than three. They are also called determinants with the following denotations

$$A = \begin{pmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{12} & a_{12} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{pmatrix}$$

$$\det(A) = \det \begin{pmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{12} & a_{12} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{pmatrix} = |A| = \begin{vmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{12} & a_{12} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{vmatrix}$$

Thus the determinant of a square  $n$  by  $n$  matrix  $A$  is a number. To define determinants of orders higher than 3, we have to introduce the notion of a permutation.

## **Permutation**

A permutation of the set  $P_n = \{1, 2, \mathbf{K}, n\}$  is any one-to-one mapping of  $P_n$  onto itself. It is most conveniently defined by an ordered sequence  $(i_1, i_2, \mathbf{K}, i_n)$  of the numbers from  $P_n$ .

These are, for example, all the permutations of  $P_3 = \{1, 2, 3\}$

$$(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$$

## Parity of a permutation

For a permutation  $p = (i_1, i_2, \mathbf{K}, i_n)$  we say that it has the even parity or that it is even or that  $p(p) = 1$  if an even number  $k$  of swaps can be used to transform it to the so called unity permutation

$$e = (1, 2, \dots, n).$$

A permutation with the odd parity is defined in an analogous way. For an odd permutation  $p$  we put  $p(p) = -1$

For example  $(1, 4, 3, 2)$  is odd since we have

$$(1, 4, 3, 2) \xrightarrow{\text{swap 1}} (1, 3, 4, 2) \xrightarrow{\text{swap 2}} (1, 3, 2, 4) \xrightarrow{\text{swap 3}} (1, 2, 3, 4)$$

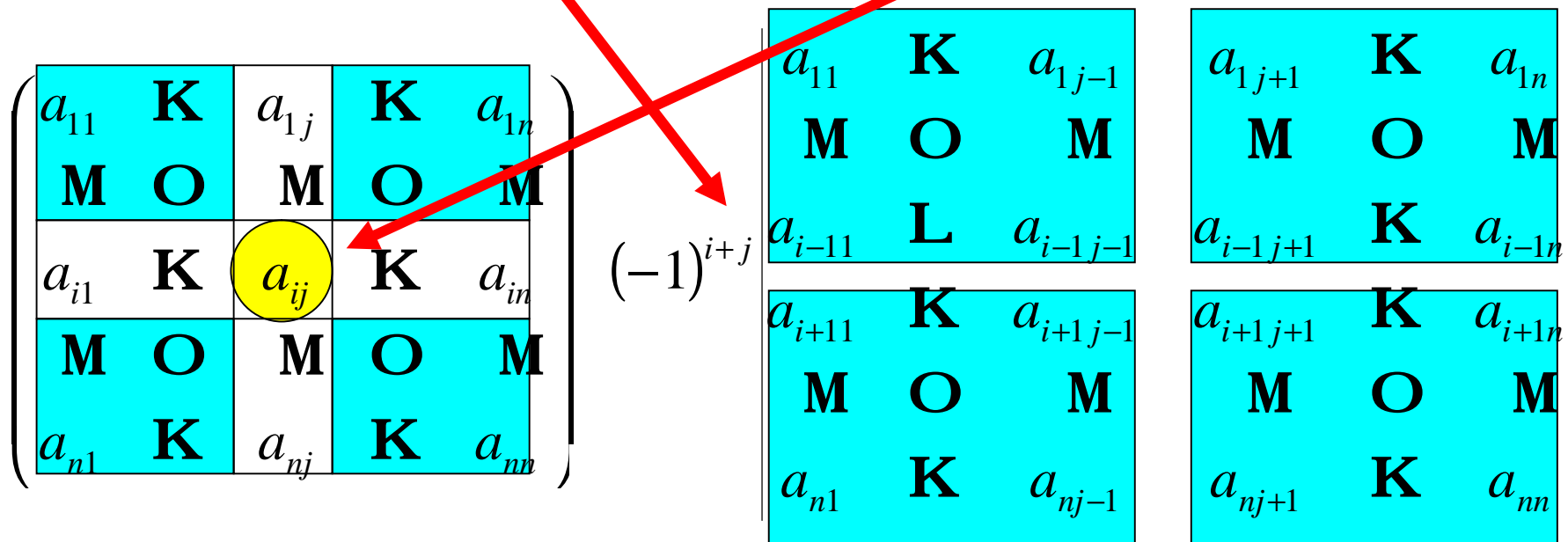
$$\boxed{\text{swap 1}} \quad \boxed{\text{swap 2}} \quad \boxed{\text{swap 3}} \quad \Rightarrow \quad p(1, 4, 3, 2) = -1$$

$$\det \begin{pmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{21} & a_{22} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn}i \end{pmatrix} = \sum_{(i_1, i_2, \mathbf{K}, i_n) \in S_n} p(i_1, i_2, \mathbf{K}, i_n) a_{1i_1} a_{2i_2} \mathbf{L} a_{ni_n}$$

The summation is performed for all the permutations of  
 $\{1, 2, \dots, n\}$

It is very impractical to calculate determinants using the definition. Several other methods exist, one of them uses the notion of an algebraic complement,

$A_{ij}$  is the algebraic complement to  $a_{ij}$



For a matrix  $A = \begin{pmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{12} & a_{22} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{pmatrix}$

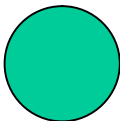
we have

$$\det A = \sum_{j=1}^n a_{ij} A_{ij} \text{ for any } i = 1, 2, \dots, n$$

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where  $a_{ij}$  is the algebraic complement to  $A_{ij}$

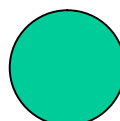
## Properties of determinants

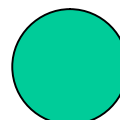


$$\begin{vmatrix} ca_{11} & ca_{12} & \mathbf{K} & ca_{1n} \\ a_{12} & a_{22} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{vmatrix} = c \begin{vmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{12} & a_{22} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{vmatrix}$$



$$\det A^T = \det A$$

 If, in a matrix, we add a linear combination of several rows (columns) to another row (column) in the matrix, the determinant of the resulting matrix remains the same.

 If we swap two rows or columns in a matrix, the resulting matrix has the same determinant but with an opposite sign

We can use these properties to calculate the determinant of a matrix. Using the above-described operation, we transform the matrix to a staircase matrix and then use the following formula:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \mathbf{K} & a_{1n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \mathbf{K} & a_{2n-1} & a_{2n} \\ 0 & 0 & a_{33} & \mathbf{K} & a_{3n-1} & a_{3n} \\ \mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ 0 & 0 & 0 & \mathbf{K} & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & 0 & \mathbf{K} & 0 & a_{nn} \end{pmatrix} = a_{11}a_{22}a_{33}\mathbf{L} a_{n-1n-1}a_{nn}$$



## Cramer's rule

$$a_{11}x_1 + a_{12}x_2 + \mathbf{L} + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \mathbf{L} + a_{2n}x_n = b_2$$

$$\mathbf{M} \quad \mathbf{M} \quad \mathbf{O} \quad \mathbf{M} \quad \mathbf{M}$$

$$a_{n1}x_1 + a_{n2}x_2 + \mathbf{L} + a_{nn}x_n = b_n$$

$$x_i = \frac{\begin{vmatrix} a_{11} & \mathbf{K} & a_{1i-1} & b_1 & a_{1i+1} & \mathbf{K} & a_{1n} \\ a_{21} & \mathbf{K} & a_{2i-1} & b_2 & a_{2i+1} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & \mathbf{K} & a_{ni-1} & b_n & a_{ni+1} & \mathbf{K} & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{21} & a_{22} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{vmatrix}}$$