Solving the following system of 2 equations with two unknowns

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

by the Gauss elimination method, we obtain

$$x = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$
$$y = \frac{a_{11}b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

provided that $a_{11}a_{22} - a_{12}a_{21} \neq 0$

The expressions in the numerators and denominators can be conveniently expressed by the following schemes:

$$a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$-$ $+$

$$b_1 a_{22} - a_{12} b_2 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$

$-$ $+$

$$a_{11} b_2 - b_1 a_{21} = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

$-$ $+$

The expressions

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_1 \end{vmatrix}$$

are referred to as the determinants of the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad A_1 = \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix} \qquad A_2 = \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_1 \end{pmatrix}$$

sometimes written as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |A_1| = \det(A_1) = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{etc.}$$

Thus, for the system

$$
\begin{aligned}
a_{11}x + a_{12}y &= b_1 \\
a_{21}x + a_{22}y &= b_2
\end{aligned}
$$

we can write

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

provided that

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

$$y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Similarly, for a system of three equations with three unknowns,

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

we could easily establish by the Gauss elimination method that, for example,

$$x = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

provided that the denominator is not equal to zero

Where

$$\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = b_1 a_{22} a_{33} + b_2 a_{32} a_{13} + b_3 a_{12} a_{23} - \\ -\left(b_3 a_{22} a_{13} + b_1 a_{32} a_{23} + b_2 a_{12} a_{33}\right)$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} a_{22} a_{33} + a_{21} a_{32} a_{13} + a_{31} a_{12} a_{23} - \\ -\left(a_{31} a_{22} a_{13} + a_{11} a_{32} a_{23} + a_{21} a_{12} a_{33}\right)$$

Sometimes we use the following auxiliary scheme

$$\begin{array}{cccc}
+ & & & \\
 & a_{11} & a_{12} & a_{13} \\
+ & & & \\
 & a_{21} & a_{22} & a_{23} \\
+ & & & \\
 & a_{31} & a_{32} & a_{33} \\
- & & & \\
 & a_{11} & a_{12} & a_{13} \\
- & & & \\
 & a_{21} & a_{22} & a_{23} \\
- & & &
\end{array}$$

The expression

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - \\ -\left(a_{31}a_{22}a_{13} + a_{11}a_{32}a_{23} + a_{21}a_{12}a_{33}\right)$$

is called the determinant of a 3 by 3 matrix and, for

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$ denoted by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \det(A) = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Similar schemes can also be devised for calculating systems of linear algebraic equations of orders higher than three. They are also called determinants with the following denotations

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{12} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\det(A) = \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{12} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{12} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Thus the determinant of a square *n* by *n* matrix *A* is a number. To define determinants of orders higher than 3, we have to introduce the notion of a permutation.

Permutation

A permutation of the set $P_n = \{1, 2, \ldots, n\}$ is any one-to-one mapping of P_n onto itself. It is most conveniently defined by an ordered sequence $(i_1, i_2, \ldots, i_n)$ of the numbers from P_n.

These are, for example, all the permutations of $P_3 = \{1,2,3\}$

$$(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)$$

Parity of a permutation

For a permutation $p = (i_1, i_2, \mathbf{K}, i_n)$ we say that it has the even parity or that it is even or that $p(p) = 1$ if an even number k of swaps can be used to transform it to the so called unity permutation

$e = (1,2, ..., n)$.

A permutation with the odd parity is defined in an analogous way. For an odd permutation p we put $p(p) = -1$

For example (1,4,3,2) is odd since we have

$$(1,4,3,2) \to (1,3,4,2) \to (1,3,2,4) \to (1,2,3,4)$$

swap 1   swap 2   swap 3 $\Rightarrow$ $p(1,4,3,2) = -1$

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} i \end{pmatrix} = \sum_{(i_1, i_2, \cdots, i_n) \in S_n} p\,(i_1, i_2, \cdots, i_n)\, a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

The summation is performed for all the permutations of $\{1, 2, \ldots, n\}$

It is very impractical to calculate determinants using the definition. Several other methods exist, one of them uses the notion of an algebraic complement,

A_{ij} is the algebraic complement to a_{ij}

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \qquad (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1\,j-1} & a_{1\,j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1\,1} & \cdots & a_{i-1\,j-1} & a_{i-1\,j+1} & \cdots & a_{i-1\,n} \\ a_{i+1\,1} & \cdots & a_{i+1\,j-1} & a_{i+1\,j+1} & \cdots & a_{i+1\,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

For a matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

we have

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} \text{ for any } i = 1, 2, \ldots, n$$

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij} \text{ for any } j = 1, 2, \ldots, n$$

where a_{ij} is the algebraic complement to A_{ij}

Properties of determinants

- $$\begin{vmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = c\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

- $\det A^T = \det A$

- If, in a matrix, we add a linear combination of several rows (columns) to another row (column) in the matrix, the determinant of the resulting matrix remains the same.

- If we swap two rows or columns in a matrix, the resulting matrix has the same determinant but with an opposite sign

We can use these properties to calculate the determinant of a matrix. Using the above-described operation, we transform the matrix to a staircase matrix an then use the following formula:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n-1} & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix} = a_{11}a_{22}a_{33}\cdots a_{n-1n-1}a_{nn}$$

Cramer's rule

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_1
\end{array}
$$

$$
x_i = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1i-1} & b_1 & a_{1i+1} & \cdots & a_{nn} \\ a_{21} & \cdots & a_{2i-1} & b_2 & a_{2i+1} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni-1} & b_n & a_{ni+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}
$$