

Solving the following system of 2 equations with two unknowns

$$a_{11}x + a_{12}y = b_1$$

$$a_{21}x + a_{22}y = b_2$$

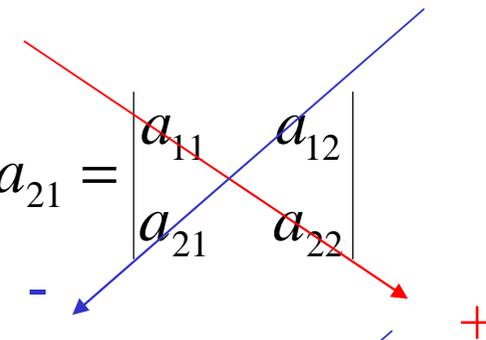
by the Gauss elimination method, we obtain

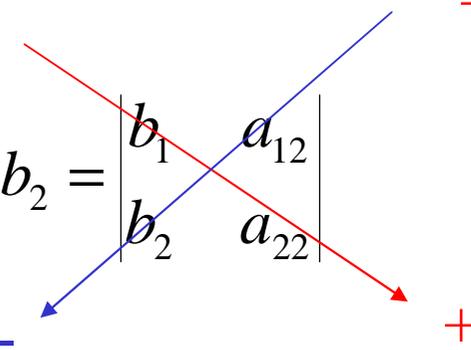
$$x = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}}$$

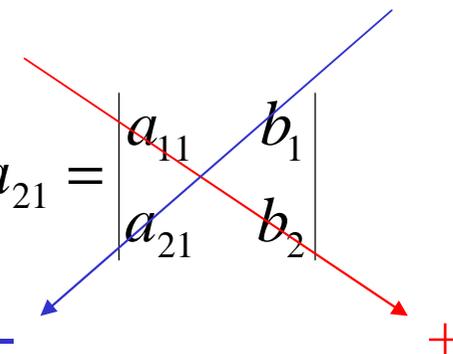
$$y = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}$$

provided that $a_{11} a_{22} - a_{12} a_{21} \neq 0$

The expressions in the numerators and denominators can be conveniently expressed by the following schemes:

$$a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$


$$b_1a_{22} - a_{12}b_2 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$


$$a_{11}b_2 - b_1a_{21} = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$


The expressions

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \quad \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_1 \end{vmatrix}$$

are referred to as the determinants of the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad A_1 = \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix} \quad A_2 = \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_1 \end{pmatrix}$$

sometimes written as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |A_1| = \det(A_1) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{etc.}$$

Thus, for the system

$$a_{11}x + a_{12}y = b_1$$

$$a_{21}x + a_{22}y = b_2$$

we can write

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

provided that

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

Similarly, for a system of three equations with three unknowns,

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

we could easily establish by the Gauss elimination method that, for example,

$$x = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

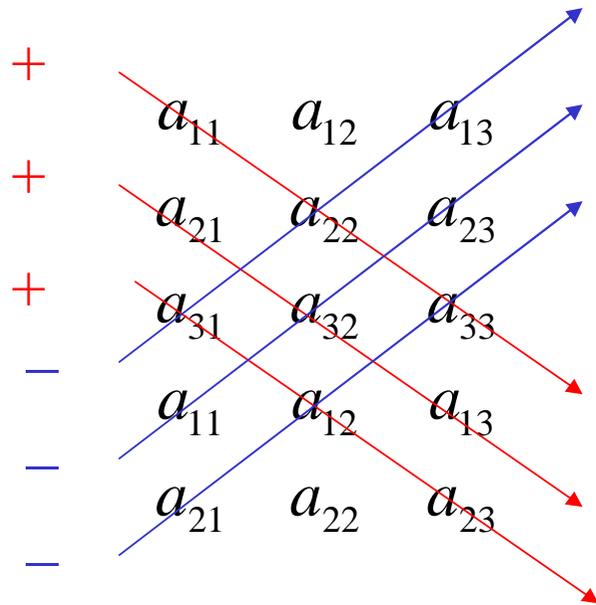
provided that the denominator is not equal to zero

Where

$$\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = b_1 a_{22} a_{33} + b_2 a_{32} a_{13} + b_3 a_{12} a_{23} - \\ - (b_3 a_{22} a_{13} + b_1 a_{32} a_{23} + b_2 a_{12} a_{33})$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} a_{22} a_{33} + a_{21} a_{32} a_{13} + a_{31} a_{12} a_{23} - \\ - (a_{31} a_{22} a_{13} + a_{11} a_{32} a_{23} + a_{21} a_{12} a_{33})$$

Sometimes we use the following auxiliary scheme



The expression

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - \\ - (a_{31}a_{22}a_{13} + a_{11}a_{32}a_{23} + a_{21}a_{12}a_{33})$$

is called the determinant of a 3 by 3 matrix and, for

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{denoted by}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \det(A) = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Similar schemes can also be devised for calculating systems of linear algebraic equations of orders higher than three. They are also called determinants with the following denotations

$$A = \begin{pmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{12} & a_{12} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{pmatrix}$$

$$\det(A) = \det \begin{pmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{12} & a_{12} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{pmatrix} = |A| = \begin{vmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{12} & a_{12} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{vmatrix}$$

Thus the determinant of a square n by n matrix A is a number. To define determinants of orders higher than 3, we have to introduce the notion of a permutation.

Permutation

A permutation of the set $P_n = \{1, 2, \mathbf{K}, n\}$ is any one-to-one mapping of P_n onto itself. It is most conveniently defined by an ordered sequence $(i_1, i_2, \mathbf{K}, i_n)$ of the numbers from P_n .

These are, for example, all the permutations of $P_3 = \{1, 2, 3\}$

$(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$

Parity of a permutation

For a permutation $p = (i_1, i_2, \mathbf{K}, i_n)$ we say that it has the even parity or that it is even or that $p(p) = 1$ if an even number k of swaps can be used to transform it to the so called unity permutation

$$e = (1, 2, \dots, n).$$

A permutation with the odd parity is defined in an analogous way. For an odd permutation p we put $p(p) = -1$

For example $(1, 4, 3, 2)$ is odd since we have

$$(1, 4, 3, 2) \xrightarrow{\text{swap 1}} (1, 3, 4, 2) \xrightarrow{\text{swap 2}} (1, 3, 2, 4) \xrightarrow{\text{swap 3}} (1, 2, 3, 4)$$

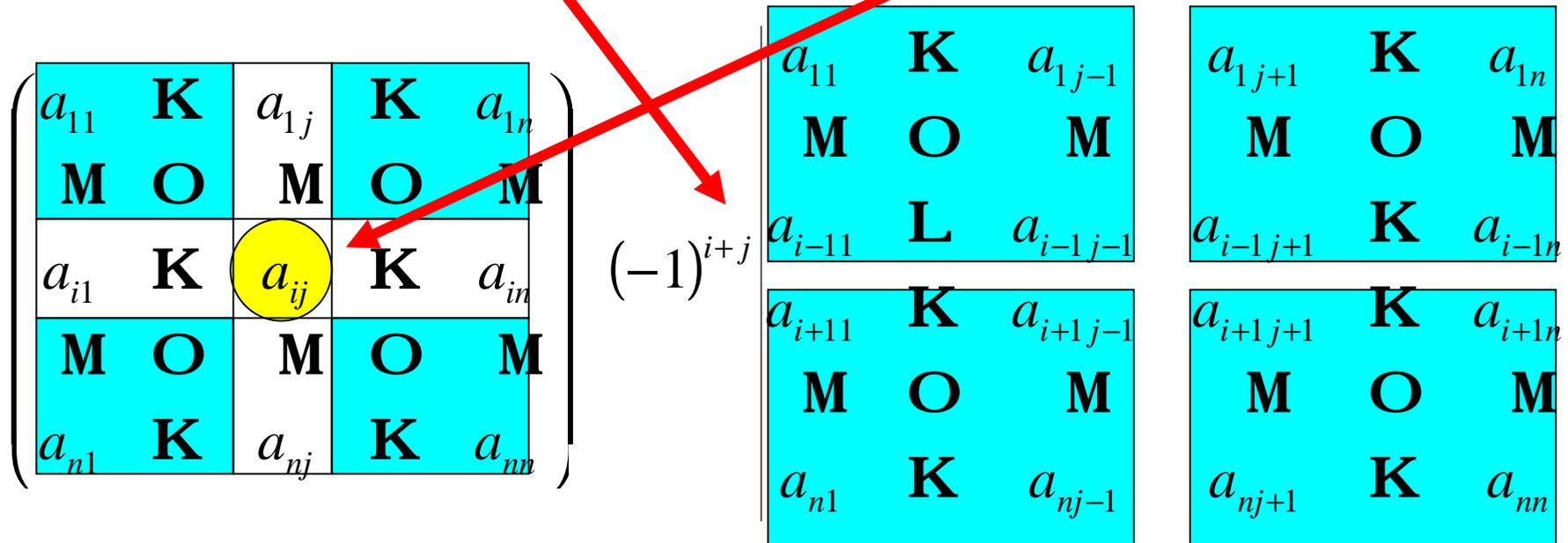
$$\boxed{\text{swap 1}} \quad \boxed{\text{swap 2}} \quad \boxed{\text{swap 3}} \quad \rightarrow \quad p(1, 4, 3, 2) = -1$$

$$\det \begin{pmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{21} & a_{22} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn}i \end{pmatrix} = \sum_{(i_1, i_2, \mathbf{K}, i_n) \in S_n} p(i_1, i_2, \mathbf{K}, i_n) a_{1i_1} a_{2i_2} \mathbf{L} a_{ni_n}$$

The summation is performed for all the permutations of
 $\{1, 2, \dots, n\}$

It is very impractical to calculate determinants using the definition. Several other methods exist, one of them uses the notion of an algebraic complement,

A_{ij} is the algebraic complement to a_{ij}



For a matrix $A = \begin{pmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{12} & a_{22} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{pmatrix}$

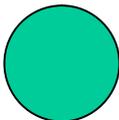
we have

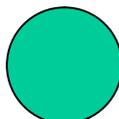
$$\det A = \sum_{j=1}^n a_{ij} A_{ij} \text{ for any } i = 1, 2, \dots, n$$

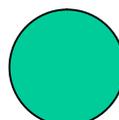
$$\det A = \sum_{i=1}^n a_{ij} A_{ij} \text{ for any } j = 1, 2, \dots, n$$

where a_{ij} is the algebraic complement to A_{ij}

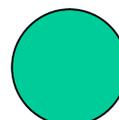
Properties of determinants


$$\begin{vmatrix} ca_{11} & ca_{12} & \mathbf{K} & ca_{1n} \\ a_{12} & a_{22} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{vmatrix} = c \begin{vmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{12} & a_{22} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{vmatrix}$$


$$\det A^T = \det A$$



If, in a matrix, we add a linear combination of several rows (columns) to another row (column) in the matrix, the determinant of the resulting matrix remains the same.



If we swap two rows or columns in a matrix, the resulting matrix has the same determinant but with an opposite sign

We can use these properties to calculate the determinant of a matrix. Using the above-described operation, we transform the matrix to a staircase matrix and then use the following formula:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \mathbf{K} & a_{1n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \mathbf{K} & a_{2n-1} & a_{2n} \\ 0 & 0 & a_{33} & \mathbf{K} & a_{3n-1} & a_{3n} \\ \mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ 0 & 0 & 0 & \mathbf{K} & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & 0 & \mathbf{K} & 0 & a_{nn} \end{pmatrix} = a_{11} a_{22} a_{33} \mathbf{L} a_{n-1n-1} a_{nn}$$

Cramer's rule

$$a_{11}x_1 + a_{12}x_2 + \mathbf{L} + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \mathbf{L} + a_{2n}x_n = b_2$$

$$\mathbf{M} \quad \mathbf{M} \quad \mathbf{O} \quad \mathbf{M} \quad \mathbf{M}$$

$$a_{n1}x_1 + a_{n2}x_2 + \mathbf{L} + a_{nn}x_n = b_n$$

$$x_i = \frac{\begin{vmatrix} a_{11} & \mathbf{K} & a_{1i-1} & b_1 & a_{1i+1} & \mathbf{K} & a_{1n} \\ a_{21} & \mathbf{K} & a_{2i-1} & b_2 & a_{2i+1} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & \mathbf{K} & a_{ni-1} & b_n & a_{ni+1} & \mathbf{K} & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \mathbf{K} & a_{1n} \\ a_{21} & a_{22} & \mathbf{K} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{K} & a_{nn} \end{vmatrix}}$$