Example – set of integers

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

for every

$$i + j = k$$

pair of integers

resulting integer

operation of addition

$$(a+b)+c = a+(b+c)$$

addition is an associative operation

$$0 + a = a + 0 = a$$

Z contains 0 as a neutral element

$$a + (-a) = (-a) + a = 0$$

an inverse (opposite) integer exists to each integer that added results in zero

Example – set of rational numbers

$$Q = \left\{ \mathbf{K}\, \frac{-14}{27}, \mathbf{K}, \frac{1}{7}, \mathbf{K}, \frac{238}{523}, \mathbf{K} \right\}$$

for every

$$p : q = r$$

pair of rational numbers

operation of multiplication

resulting rational

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

multiplication is an associative operation

$$1 \cdot a = a \cdot 1 = a$$

Q contains 1 as a neutral element

$$a \cdot \left(\frac{1}{a} \right) = \left(\frac{1}{a} \right) \cdot a = 1$$

an inverse rational number exists that multiplied with the original number gives zero

Example – set of regular n by n matrices

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

A is regular if $\det(A)\neq 0$

$AB=C$

operation of matrix multiplication

- $A(BC)=(AB)C$
- $AI_n=I_nA$
- $\det(A)\neq 0 \Rightarrow \exists A^{-1}: A^{-1}A=AA^{-1}=I_n$

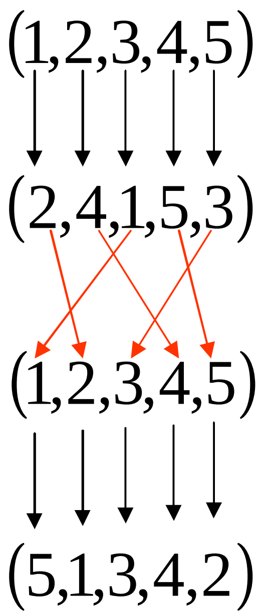
$$I_n=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

n by n unity matrix

Example – composition of permutations

Let S_n denote the set of all permutations over a finite set E. We will define an operation $\circ$ of composing two permutations in S_n.

Let for example $n = 5$, $E = \{1,2,3,4,5\}$, and let us consider two permutations, say, $p_1 = (2,4,1,5,3)$ and $p_2 = (5,1,3,4,2)$



$$\begin{array}{c}(1,2,3,4,5)\\ \downarrow\downarrow\downarrow\downarrow\downarrow \\ p_3 = (1,4,5,2,3)\end{array}$$

By first applying permutation p_1 and subsequently p_2 we obtain permutation p_3, this is denoted

$p_3 = p_1 \circ p_2$

operation of composition of two permutations

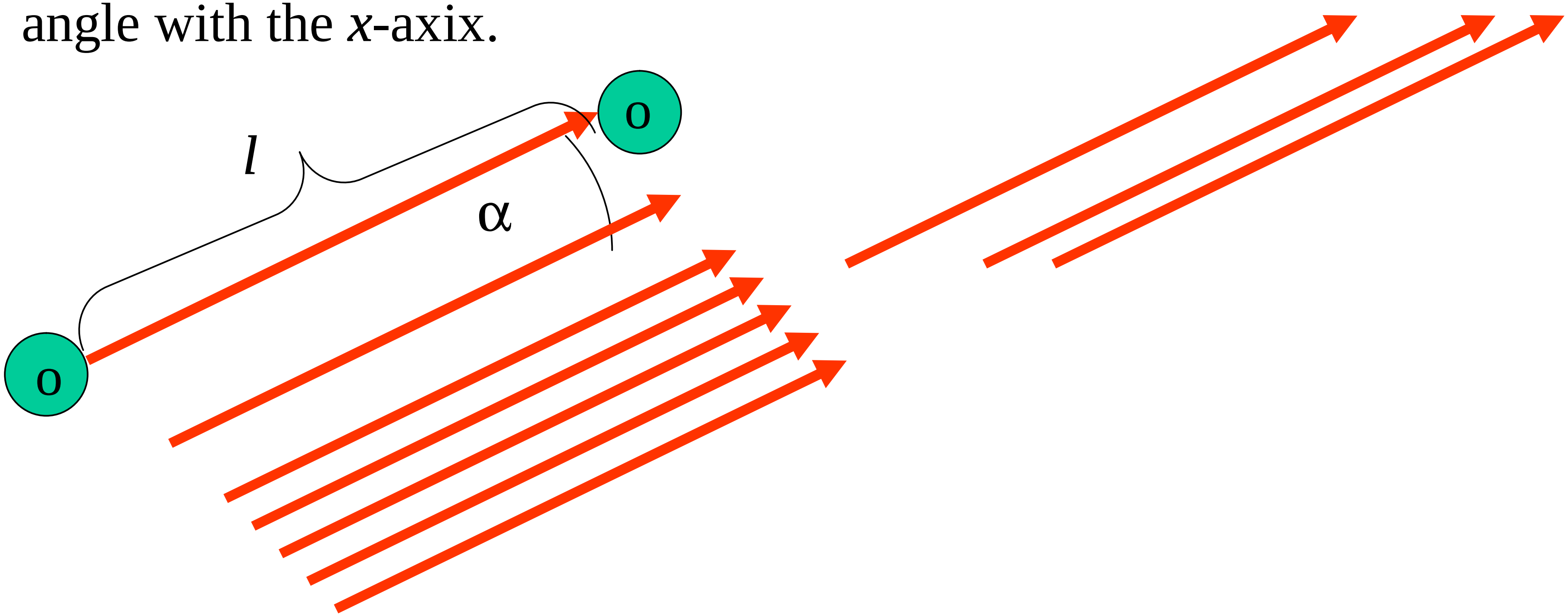
We could easily prove the following properties of the composition of permutations. For S_n and an operation $\circ$ of composition defined, we have, for every $\pi_1, \pi_2, \pi_3 \in S_n$

- $(\pi_1 \circ \pi_2) \circ \pi_3 = \pi_1 \circ (\pi_2 \circ \pi_3)$
- $\exists e \in S_n : \forall \pi \in S_n : e \circ \pi = \pi \circ e = \pi$
- $\forall \pi \in S_n : \exists \pi^{-1} \in S_n : \pi \circ \pi^{-1} = \pi^{-1} \circ \pi = e$

here $e = (1, 2, \ldots, n)$

Example - displacements of an object in space

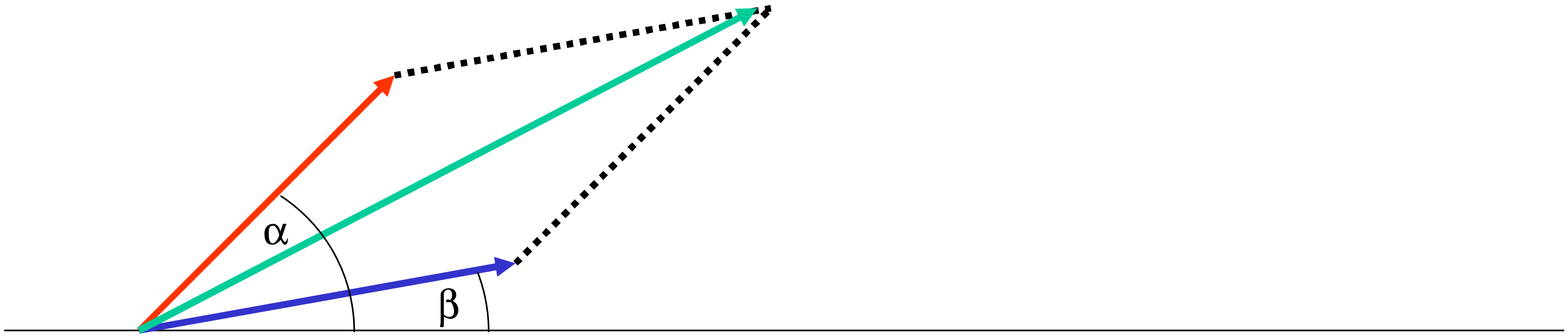
Let *D* be a set of the displacements of an object in a plane along a finite line segment. Each such displacement $\boldsymbol{d}$ will be defined by a pair $\boldsymbol{d} = (l, a)$ where *l* is the length of the displacement and α its angle with the *x*-axix.



If we apply first a displacement $\boldsymbol{d}_1 = (l_1, \alpha_1)$ to a given object $\boldsymbol{o}$ and subsequently apply a displacement $\boldsymbol{d}_2 = (l_2, \alpha_2)$ to the new position of the object resulting from the first displacement, we may think of the final position of $\boldsymbol{o}$ as of the result of a displacement $\boldsymbol{d}_3 = (l_3, \alpha_3)$ composed of the displacements $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$. We write $\boldsymbol{d}_3 = \boldsymbol{d}_1 \oplus \boldsymbol{d}_2$

Using elementary trigonometry we can establish that

$$a_3 = \frac{a_1 + a_2}{2}, l_3 = l_1^2 + l_2^2 + 2\cos|a - b| l_1 l_2$$



Again, using trigonometry, we could prove that the following formulas hold for the operation of displacement where d_1, d_2, $d_3 \in D$ and $\boldsymbol{o}$ denotes the "stationary" displacement.

- $\left(\overrightarrow{d_1} \oplus \overrightarrow{d_2}\right) \oplus \overrightarrow{d_3} = \overrightarrow{d_1} \oplus \left(\overrightarrow{d_2} \oplus \overrightarrow{d_3}\right)$

- $\exists \vec{o} : \forall \vec{d} : \vec{o} \oplus \vec{d} = \vec{d} \oplus \vec{o} = \vec{d}$

- $\forall \vec{d} : \exists \left(-\vec{d}\right) : d \oplus \left(-\vec{d}\right) = \left(-\vec{d}\right) \oplus \vec{d}$

Group

A group G is a set with a binary operation $\bullet$ that satisfies the following axioms:

For any a, b, $c \in G$, we have

the operation $\bullet$

- $(a \bullet b) \bullet c = a \bullet (b \bullet c)$

is associative

- $\exists e \in G : \forall a \in G : e \bullet a = a \bullet e = a$

has a unit

- $\forall a \in G : \exists a^{-1} \in G : a \bullet a^{-1} = a^{-1} \bullet a = e$

every element has an inverse

In addition, some groups are ***commutative***, which means that for every a, $b \in G$ we have

$$a \bullet b = b \bullet a$$

Such groups are also referred to as Abelian. The group of natural numbers with addition as the binary operation and 0 as the unit, the group of rational or real numbers without zero with multiplication as the binary operation and 1 as the unit and the group of displacements with the composition as the binary operation and "stationary" movement as the unit are all examples of Abelian groups.

Non-Abelian groups are for example the group of matrices with multiplication or the group S_n (called symmetric) of permutations with composition as the binary operation.