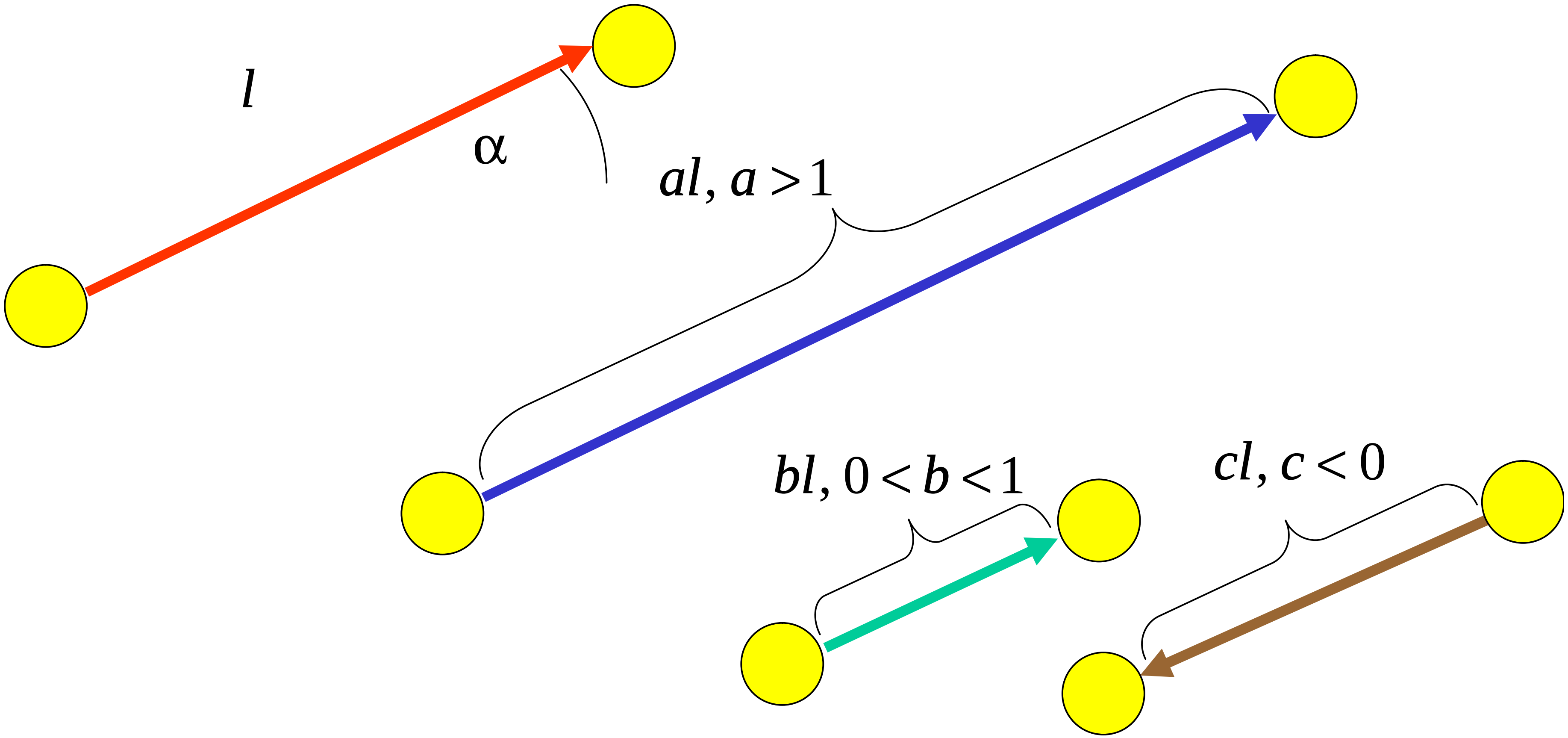
Extending a displacement

A displacement $\boldsymbol{d}$ defined by a pair $\boldsymbol{d} = (l, a)$ where l is the length of the displacement and α the angle between its direction and the *x*-axix can be "extended" by multiplying its "distance"



Thus any real number ***a*** defines an action consisting in extending the length of every displacement by ***a***.

displacement

a new displacement

$$\boldsymbol{d} = (l, a),\ \boldsymbol{a} \in R \Rightarrow \boldsymbol{a} \cdot \boldsymbol{d} = (al, a)$$

"extension" action

Properties of the extension action

For any displacements $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}$ and real numbers a, b, we can easily prove

$$a \cdot (\boldsymbol{d}_1 \oplus \boldsymbol{d}_2) = a \cdot \boldsymbol{d}_1 \oplus a \cdot \boldsymbol{d}_2$$

$$(a \oplus b) \cdot \boldsymbol{d} = a \cdot \boldsymbol{d} \oplus b \cdot \boldsymbol{d}$$

$$a \cdot (b \cdot \boldsymbol{d}) = (ab) \cdot \boldsymbol{d}$$

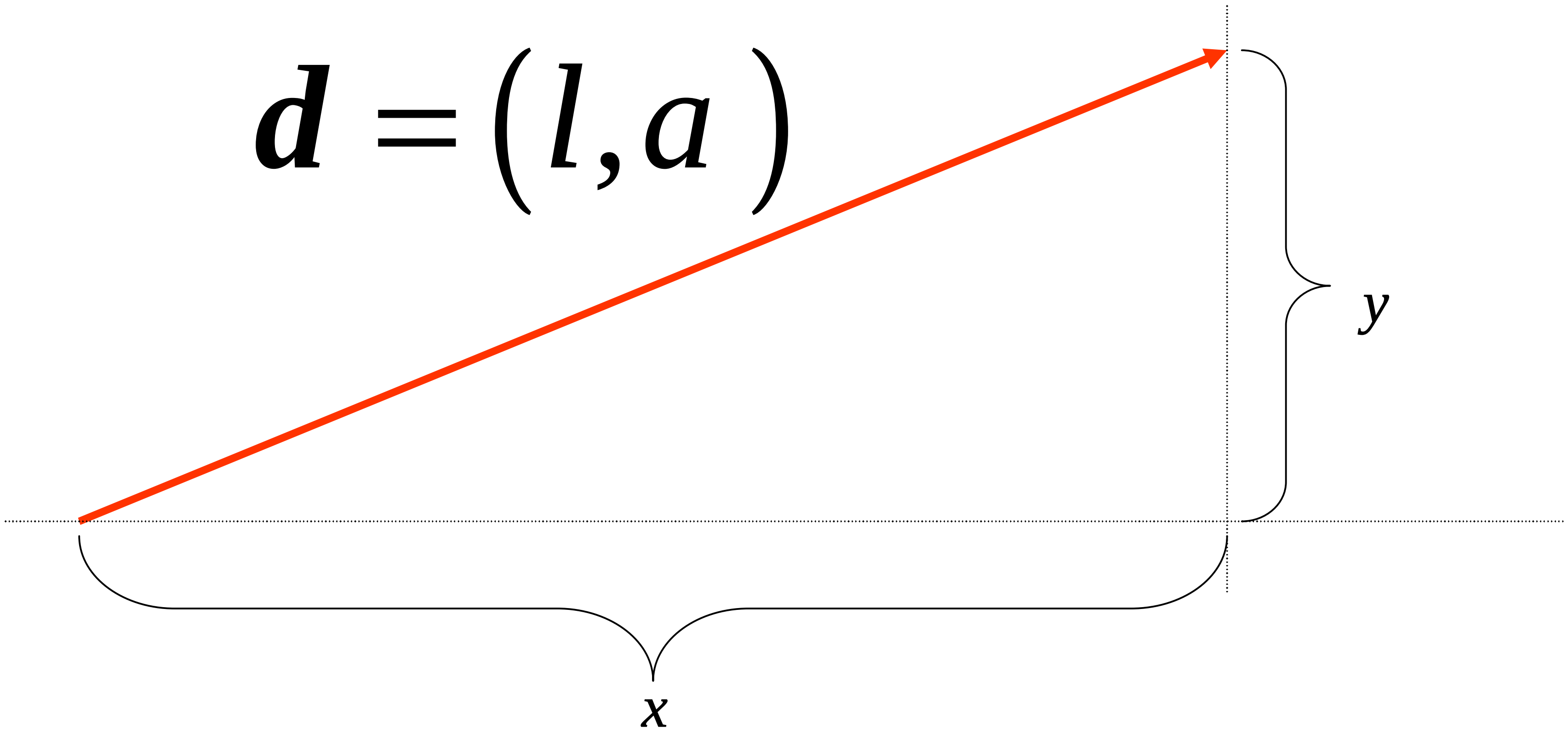
$$1 \cdot \boldsymbol{d} = \boldsymbol{d}$$

$$0 \cdot \boldsymbol{d} = \boldsymbol{o}$$

stationary displacement

<u>Another method of defining a displacement</u>

$$\boldsymbol{d} = (l, a)$$



$$\boldsymbol{d} = (x, y)$$

Thus the set of all the displacements in a plane can be viewed as the set of all pairs (x, y) of real numbers. It is easy to prove that the operation $\oplus$ of composition of two displacements and that of an extension can be defined as follows

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$a \cdot (x, y) = (ax, ay)$$

Vector space

A vector space $(V,\oplus,\cdot)$ is a set V which, together with the binary operation $\oplus$, forms an Abelian group and $\cdot$ is a mapping

$$\cdot : R\times V \to V$$

such that, for every $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3 \in V,\ a,b\in R$, we have

$$a\cdot(\boldsymbol{v}_1 \oplus \boldsymbol{v}_2) = a\cdot\boldsymbol{v}_1 \oplus a\cdot\boldsymbol{v}_2$$

$$(a+b)\cdot\boldsymbol{v} = a\cdot\boldsymbol{v} \oplus b\cdot\boldsymbol{v}$$

$$a\cdot(b\cdot\boldsymbol{v}) = (ab)\cdot\boldsymbol{v}$$

$$1\cdot\boldsymbol{v} = \boldsymbol{v}$$

Conventions

- Sometimes we will just write V instead of $(V,\oplus,\cdot)$

- The unity element in the Abelian group is called the zero vector.

- Save in cases where this might cause confusion, we will write + instead of $\oplus$

- Save in cases where this might cause confusion, we will leave out the sign $\bullet$ for the operation of extension

- The elements of a vector space V are called vectors and denoted by letters with arrows (as opposed to the real numbers, which are called scalars here).

<u>Linear combination of vectors</u>

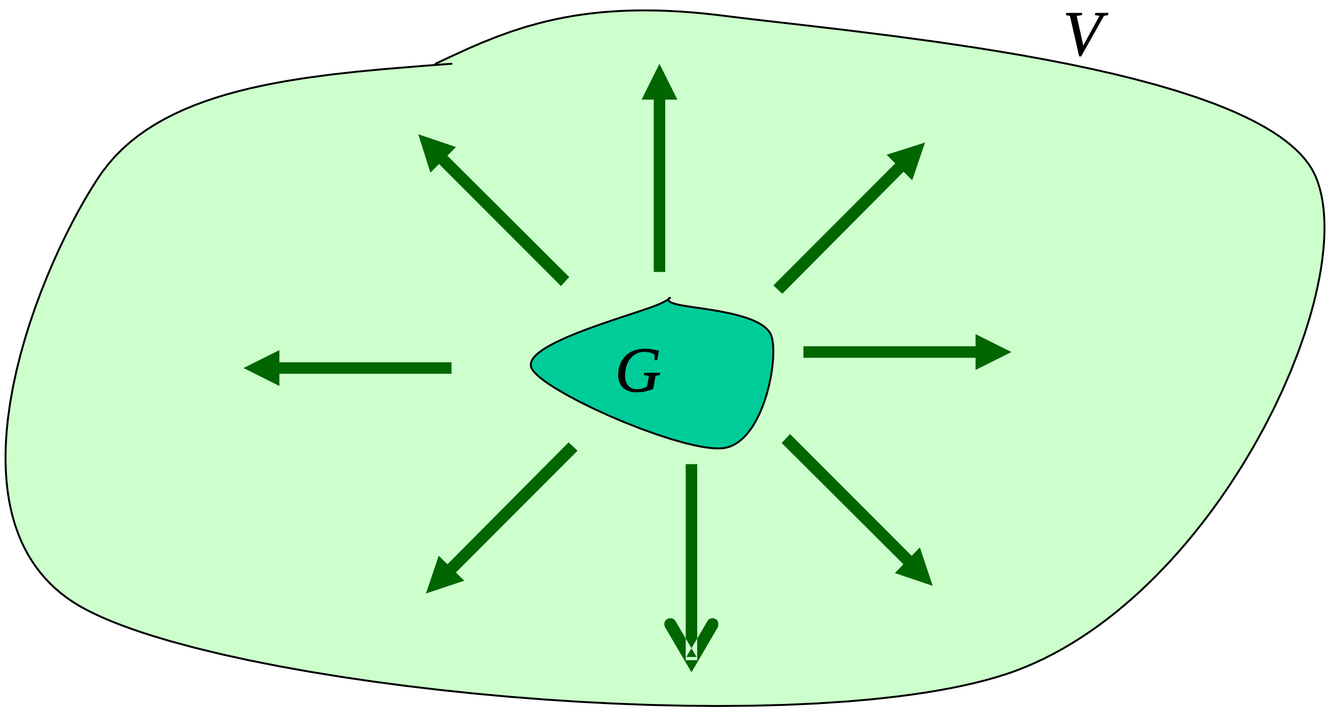
Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n \in V$ and $c_1, c_2, \ldots, c_n \in R$.

If $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$, we say that $\boldsymbol{v}$ is a linear combination of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n \in V$

By $\mathrm{Lin}\left(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\right)$ we will denote the set of all linear combinations of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$

Generating set

Let V be a vector space and G a finite subset of V. If, for every, $\boldsymbol{v} \in V$ we have $\boldsymbol{v} \in \operatorname{Lin}(G)$, then we say that G is a generating set for V or that G generates V.



Linearly independent vectors

Let V be a vector space and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n \in V$

If
$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{o} \Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

we say that vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly independent.

Otherwise we say that they are linearly dependent.

If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n \in V$ are linearly dependent,

we can assume that, in the expression $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{o}$,

say, $c_1 \neq 0$. Then

$$\boldsymbol{v}_1 = -\frac{c_2}{c_1}\boldsymbol{v}_2 - -\frac{c_3}{c_1}\boldsymbol{v}_3 - \cdots - -\frac{c_n}{c_1}\boldsymbol{v}_n$$

and thus $\boldsymbol{v}_1$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$

Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n \in V$ are linearly independent if and only if none of the vectors is a linear combination of the others.

A basis of a vector space

Let V be a vector space and B its finite subset. We say that B is a basis of V if B is a linearly independent generating set.

Bases of a vector space have the following properties:

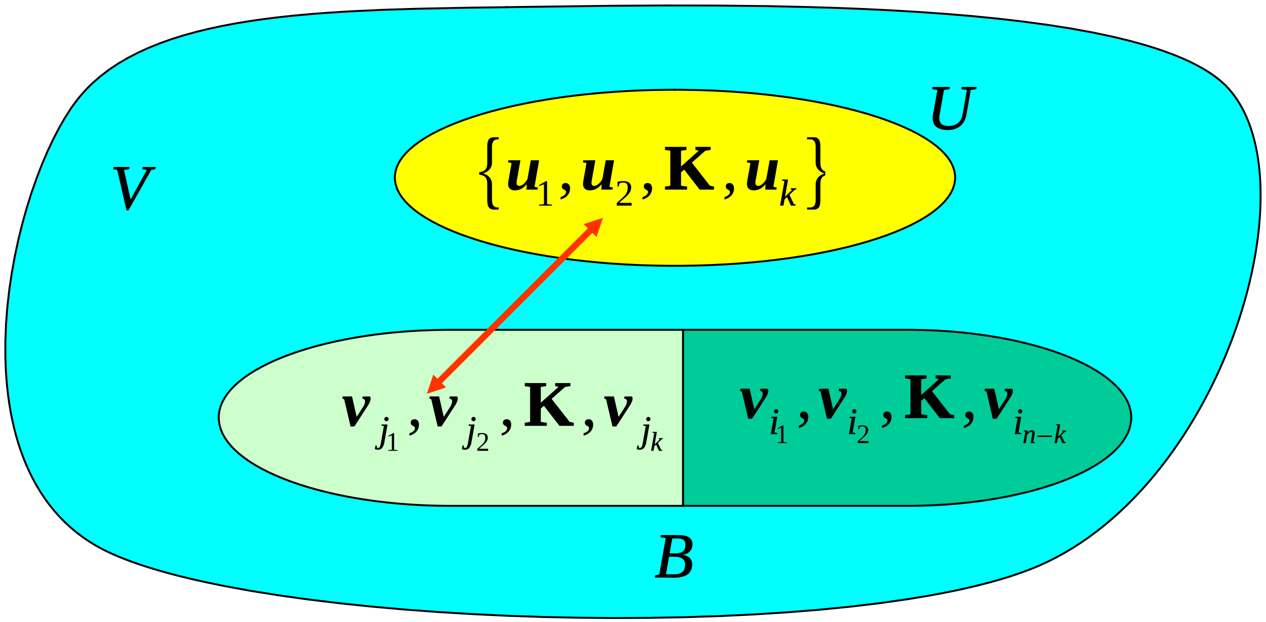
1. Every two bases of a vector space have the same number of vectors
2. Every linearly independent subset of vectors can be completed to form a basis
3. Every basis is a maximal independent set
4. Every basis is a minimal generating set

The properties of the bases of a vector space can be proved using the following <u>**Steinitz theorem**</u>:

Let $B=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_n\}$ be a basis of a vector space V.

Let $U=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ be an independent set of vectors in V.

Then $k\le n$ and B contains $(n - k)$ vectors $\boldsymbol{v}_{i_1},\boldsymbol{v}_{i_2},\ldots,\boldsymbol{v}_{i_{n-k}}$ such that the set $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k,\boldsymbol{v}_{i_1},\boldsymbol{v}_{i_2},\ldots,\boldsymbol{v}_{i_{n-k}}\}$ is a basis in V.



To prove the Steinitz theorem we can use the following **<u>exchange lemma</u>**

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \mathbf{K}, \boldsymbol{v}_n$ be a basis in V. For any non-zero vector $\boldsymbol{u} \in V$ there exists a vector $\boldsymbol{v}_i$ such that $\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \mathbf{K}, \boldsymbol{v}_{i-1}, \boldsymbol{u}, \boldsymbol{v}_{i+1}, \mathbf{K}, \boldsymbol{v}_n \}$ is a basis in V.

Dimension of a vector space

If a vector space V has a basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ we say that V has dimension n or that V is an n-dimensional vector space.

The above definition is correct since any two bases have the same number of vectors.

Let V be a vector space with a dimension n and let $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ be its basis. Then, for every $\boldsymbol{u} \in V$, there exists a **unique** set of real numbers $\{u_1, u_2, \ldots, u_n\}$ such that $\boldsymbol{u} = u_1\boldsymbol{b}_1 + u_2\boldsymbol{b}_2 + \cdots + u_n\boldsymbol{b}_n$

The sequence $(u_1, u_2, \ldots, u_n)$ is called the **coordinates** of vector $\boldsymbol{u}$ in basis B.

An n-dimensional vector space V may be identified with the vector space of all n-tuples of real numbers $(u_1, u_2, ..., u_n)$ with the operations $\oplus$ and • defined as

- $\left(u_1, u_2, \mathbf{K}, u_n\right) \oplus \left(v_1, v_2, \mathbf{K}, v_n\right) = \left(u_1 + v_1, u_2 + v_2, \mathbf{K}, u_n + v_n\right)$
- $a \cdot \left(u_1, u_2, \mathbf{K}, u_n\right) = \left(au_1, au_2, \mathbf{K}, au_n\right)$