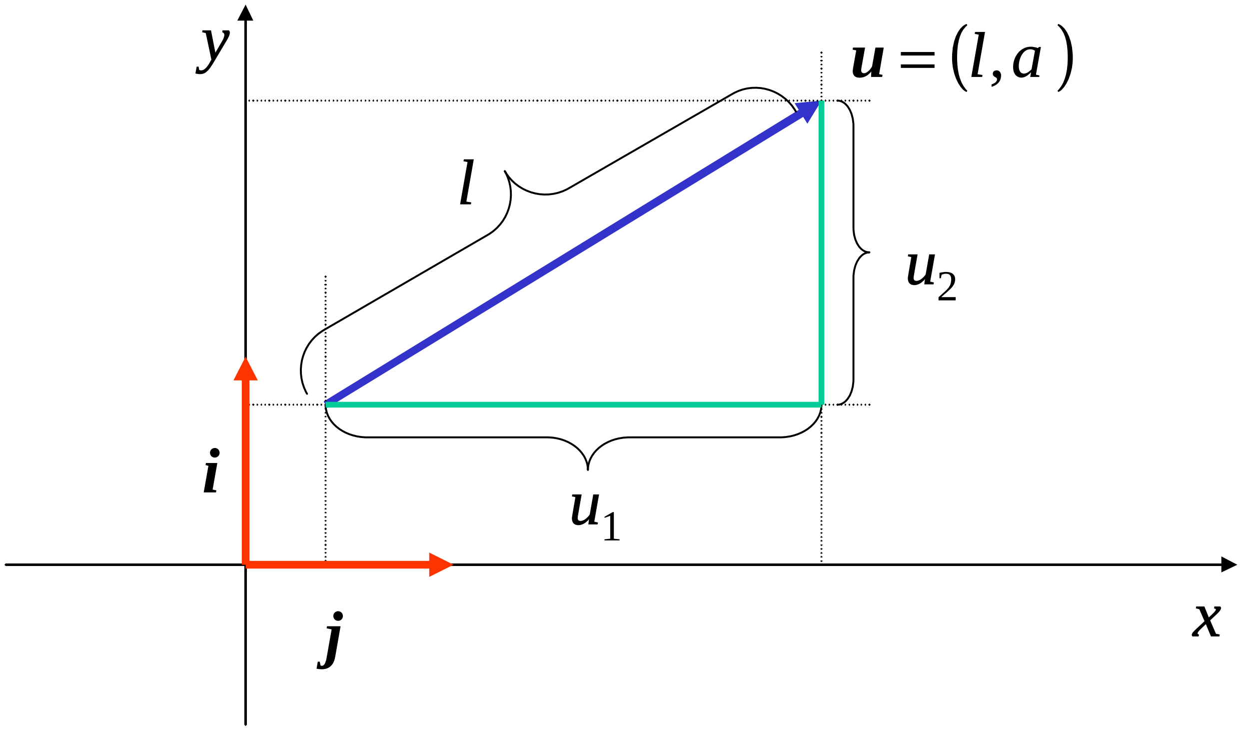
Vector space V_2 of displacements in a plane

$$\boldsymbol{i} = (1,0)$$

$$\boldsymbol{j} = \left(1, \frac{p}{2}\right)$$



$$\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$$

Using the basis $\{\boldsymbol{i}, \boldsymbol{j}\}$, we can identify the vector space V_2 of displacements in a plane with the vector space of all the pairs (u_1, u_2) of real numbers

Vector space V_3 of 3D displacements

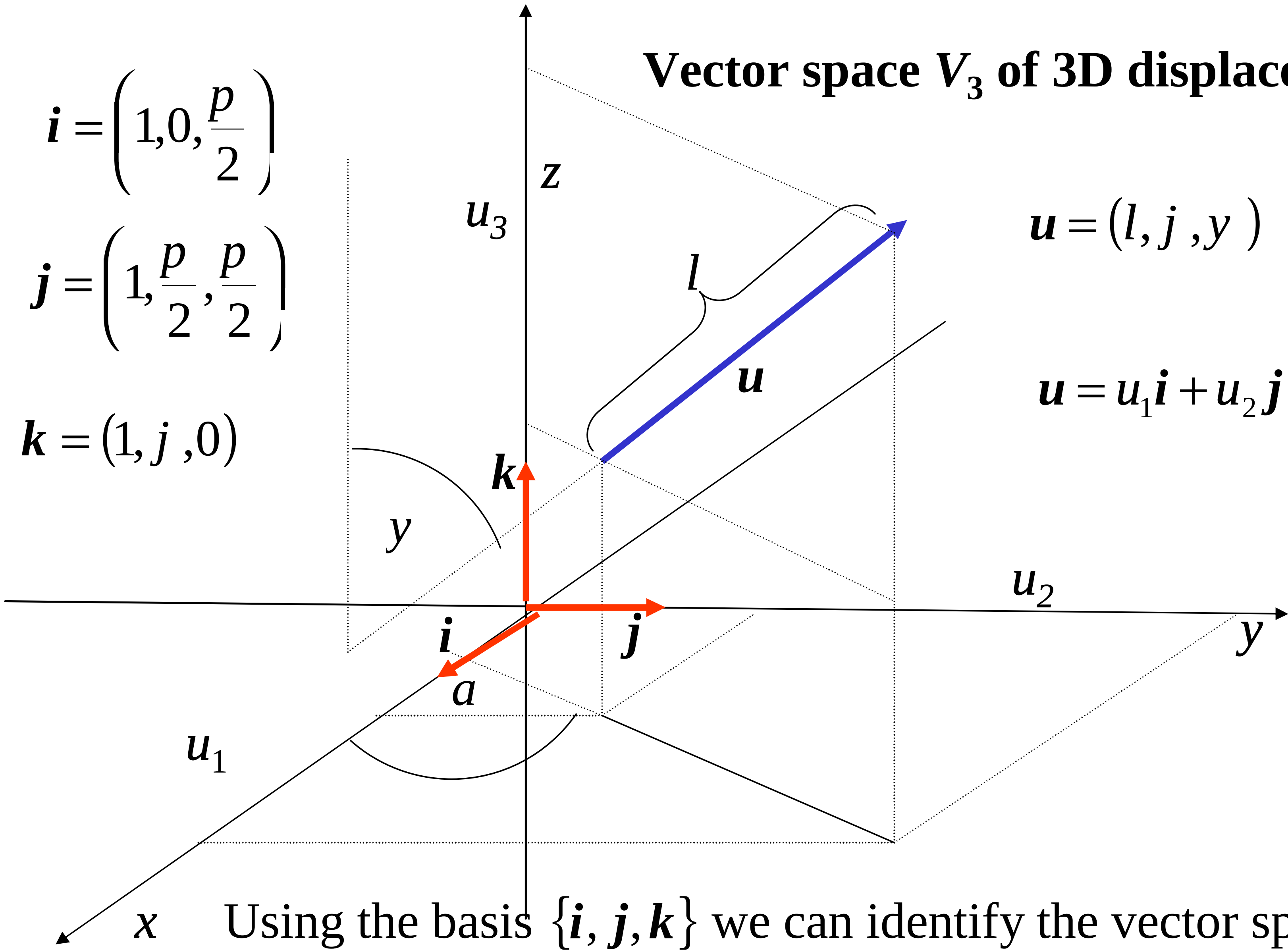
$$\boldsymbol{i} = \left(1, 0, \frac{p}{2}\right)$$

$$\boldsymbol{j} = \left(1, \frac{p}{2}, \frac{p}{2}\right)$$

$$\boldsymbol{k} = (1, j, 0)$$

$$\boldsymbol{u} = (l, j, y)$$

$$\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$$



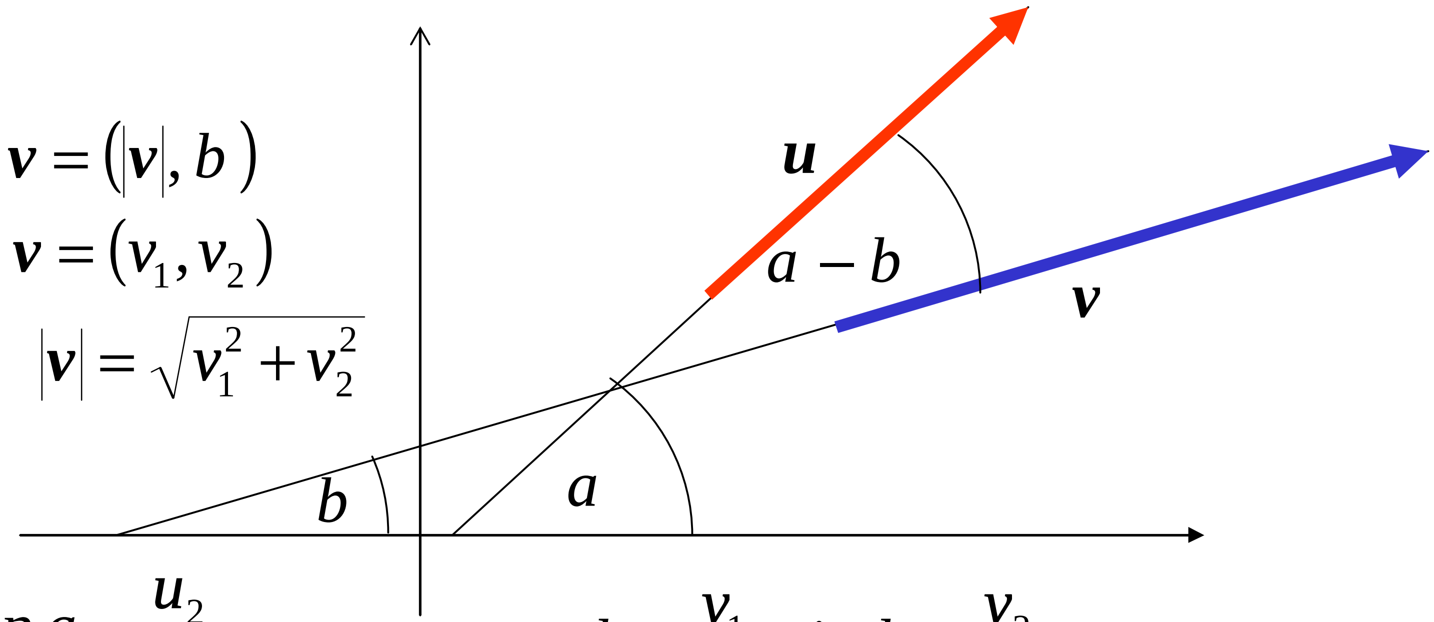
Using the basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ we can identify the vector space V_3 with the vector space of all the triples (u_1, u_2, u_3) of real numbers

The angle of two displacements in a plane

$$\boldsymbol{u} = (|\boldsymbol{u}|, a) \quad \boldsymbol{v} = (|\boldsymbol{v}|, b)$$

$$\boldsymbol{u} = (u_1, u_2) \quad \boldsymbol{v} = (v_1, v_2)$$

$$|\boldsymbol{u}| = \sqrt{u_1^2 + u_2^2} \quad |\boldsymbol{v}| = \sqrt{v_1^2 + v_2^2}$$



$$\cos a = \frac{u_1}{|\boldsymbol{u}|}, \sin a = \frac{u_2}{|\boldsymbol{u}|}$$

$$\cos b = \frac{v_1}{|\boldsymbol{v}|}, \sin b = \frac{v_2}{|\boldsymbol{v}|}$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b = \frac{u_1 v_1}{|\boldsymbol{u}||\boldsymbol{v}|} + \frac{u_2 v_2}{|\boldsymbol{u}||\boldsymbol{v}|} = \frac{u_1 v_1 + u_2 v_2}{|\boldsymbol{u}||\boldsymbol{v}|}$$

The expression $u_1v_1 + u_2v_2$ is called the **<u>scalar product</u>** of vectors $\boldsymbol{u}, \boldsymbol{v}$. It is sometimes written as $\boldsymbol{u} \cdot \boldsymbol{v}$ or just $\boldsymbol{uv}$ and also called a **<u>dot product</u>** ot **<u>inner product</u>**.

If φ is the angle of ***u*** and ***v*** the scalar product can also be expressed in terms of the lengths and angles of ***u*** and ***v***:

$$\boldsymbol{uv} = \cos j\ |\boldsymbol{u}||\boldsymbol{v}|$$

Scalar product for 3D vectors is defined in an analogous way.

For vectors or displacements $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{v} = (v_1, v_2, v_3)$ that make an angle of φ, we have

$$\boxed{\boldsymbol{u}\boldsymbol{v} = \cos j \, |\boldsymbol{u}||\boldsymbol{v}|}$$

$$\boxed{\boldsymbol{u}\boldsymbol{v} = (u_1 v_1 + u_2 v_2 + u_3 v_3)}$$

Properties of scalar products

For any vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and a scalar a, we have

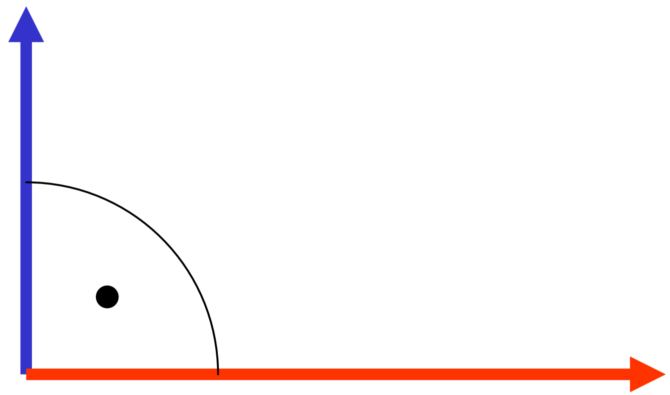
- $\boldsymbol{u}(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{u}\boldsymbol{v}+\boldsymbol{u}\boldsymbol{w}$
- $\boldsymbol{u}(a\boldsymbol{v})=a(\boldsymbol{u}\boldsymbol{v})$
- $\boldsymbol{u}\boldsymbol{u}>0$ if $\boldsymbol{u}\neq\boldsymbol{o}$
- $|\boldsymbol{u}|=\sqrt{\boldsymbol{u}\boldsymbol{u}}$

Orthogonal vectors

We say that two vectors $\boldsymbol{u}$, $\boldsymbol{v}$ are orthogonal or perpendicular to each other if

$$\boldsymbol{u}\boldsymbol{v} = 0$$

If vectors are viewed as displacements either in a plane or in a 3D space, orthogonal vectors are really perpendicular to each other in the geometric sense.



Orthonormal basis of a vector space

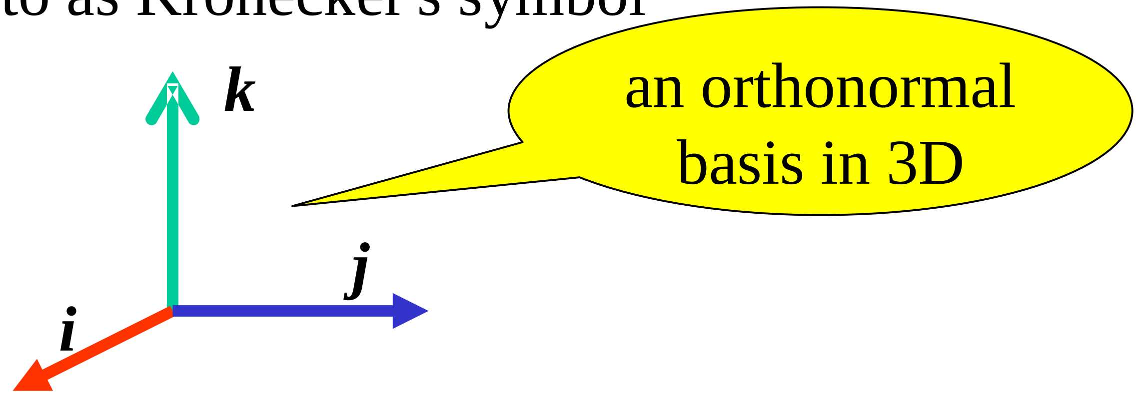
We say that $\boldsymbol{b}_1, \boldsymbol{b}_2, \mathbf{K}, \boldsymbol{b}_n$ is an orthonormal basis of a vector space V if

$$\boldsymbol{b}_i \boldsymbol{b}_j = 0 \text{ if } \boldsymbol{i} \neq \boldsymbol{j} \text{ and } \boldsymbol{b}_{\mathrm{i}} \boldsymbol{b}_i = 1$$

This is sometimes expressed in shorthand notation as

$$\boldsymbol{b}_i \boldsymbol{b}_j = d_i^{\,j}$$

where $d_i^{\,j}$ is referred to as Kronecker's symbol

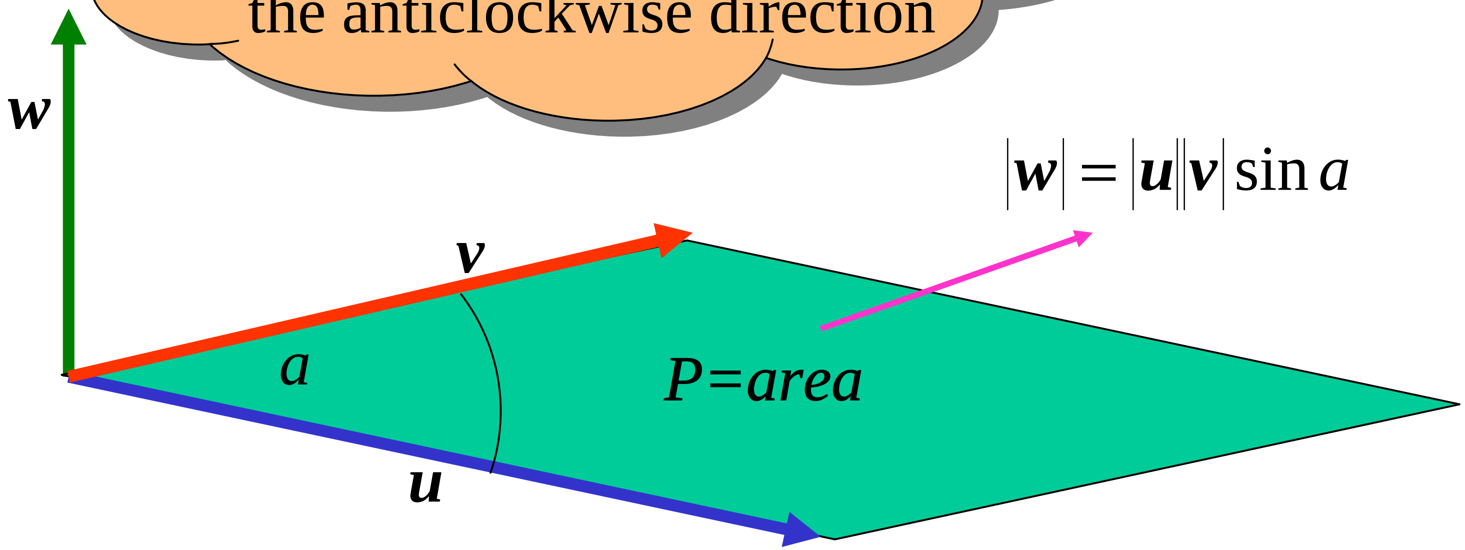


Vector product $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$

(also cross product)

only in 3D

w indicates the direction in which a screw cuts through when turned from u to v in the anticlockwise direction



$|\boldsymbol{w}| = |\boldsymbol{u}||\boldsymbol{v}|\sin a$

$\boldsymbol{wu} = \boldsymbol{wv} = 0$

Let $\boldsymbol{u} = (u_1, u_2, u_3), \boldsymbol{v} = (v_1, v_2, v_3)$, then the vector product $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$ may be expressed using the following determinant:

$$\boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \boldsymbol{i} + \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \boldsymbol{k}$$

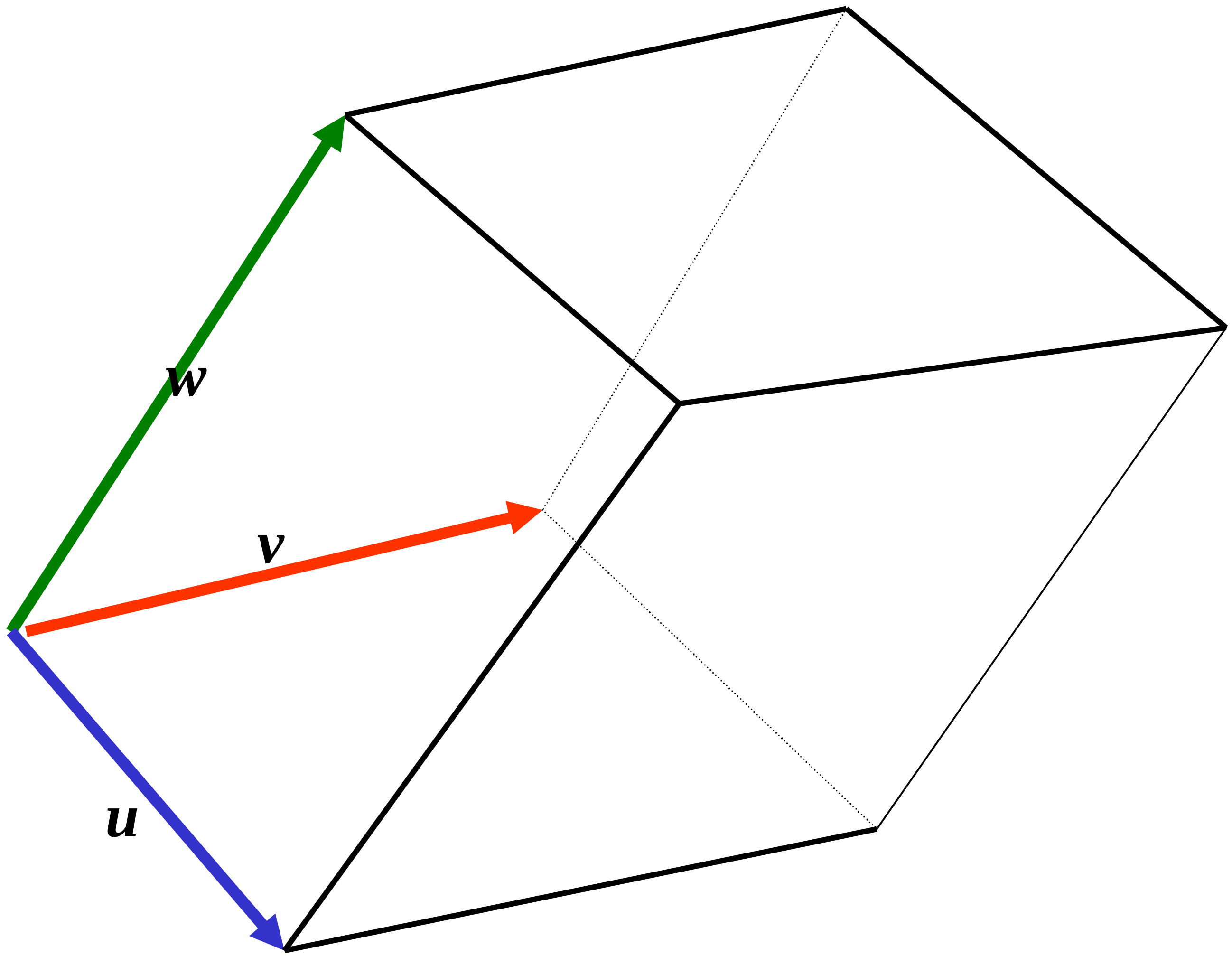
Mixed triple product

Let $\boldsymbol{u}=(u_1,u_2,u_3), \boldsymbol{v}=(v_1,v_2,v_3),, \boldsymbol{w}=(w_1,w_2,w_3)$

The expression $\boldsymbol{w}\cdot(\boldsymbol{u}\times\boldsymbol{v})$ is sometimes denoted $[\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}]$

and referred to as the mixed triple product of the vectors ***u***, ***v***, and ***w***. Sometimes it is also called **scalar triple product** or **box product**.

$$[\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}]=\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$



$\left|[\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}]\right|$ is the volume of the parallelepiped defined by the vectors $\boldsymbol{w}$, $\boldsymbol{u}$, $\boldsymbol{v}$.