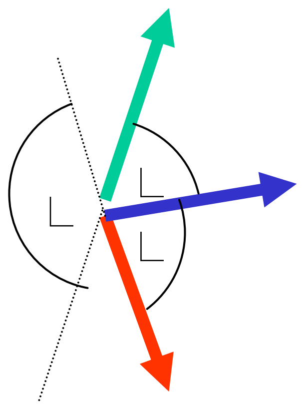
Orthonormal basis

We say that a basis $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n$ of a vector space is orthonormal if

$\boldsymbol{b}_i \cdot \boldsymbol{b}_j = 0, 1 \leq i \neq j \leq n$ the basis vectors are mutually perpendicular

$\left|\boldsymbol{b}_i\right| = 1, 1 \leq i \leq n$ the length (module, norm) of each basis vector is equal to 1



Orthonormalizing a basis

In an n-dimensional vector space V_n, for each basis $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k$, $k \le n$ of a k-dimensional vector subspace U_k of V_n, an orthonormal basis

$\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_k$ of U_k can be found.

To orthonormalize a basis $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k$, $k \le n$ we can use the following procedure:

First we find vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k$ such that $\boldsymbol{a}_i \boldsymbol{a}_j = 0$ for $i \neq j$

- $\boldsymbol{a}_1 = \boldsymbol{b}_1$

- $\boldsymbol{a}_j = \boldsymbol{b}_j - \sum_{i=1}^{j-1} c_i \boldsymbol{a}_i, \ j = 2, 3, \ldots, k$ where $c_i = \dfrac{\boldsymbol{b}_j \cdot \boldsymbol{a}_i}{\boldsymbol{a}_i \cdot \boldsymbol{a}_i}$

Then we put

$$\boldsymbol{e}_i = \frac{\boldsymbol{a}_i}{|\boldsymbol{a}_i|}, i = 1, 2, \ldots, k$$

Orthogonal matrices

A square matrix M is said to be orthogonal if $M^{-1} = M^T$,

that is, if its transpose is equal to its inverse.

Example

$$\begin{array}{c} \boldsymbol{e}_1 \\[2em] \boldsymbol{e}_2 \\[2em] \boldsymbol{e}_3 \end{array} \begin{pmatrix} \frac{3}{\sqrt{14}} & \frac{-1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{-2}{\sqrt{21}} & \frac{-4}{\sqrt{21}} & \frac{1}{\sqrt{21}} \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{14}} & \frac{-1}{\sqrt{6}} & \frac{-2}{\sqrt{21}} \\ \frac{-1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{-4}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Properties of orthogonal matrices

- If a matrix M is orthogonal then $\det(M) = \pm 1$.

- An n by n matrix M is orthogonal iff its rows (columns) viewed as coordinates of vectors form an orthonormal basis of V_n.

- Geometrically, an orthogonal matrix corresponds to a linear operator that maps an orthonormal basis to another orthonormal basis.

An orthogonal matrix M preserves scalar products:

$$\boldsymbol{a} \cdot \boldsymbol{b} = c$$

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}^T \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = c$$

viewed as two vectors

viewed as two matrices

$$M\boldsymbol{a}\cdot M\boldsymbol{b} = ?$$

M viewed as transformation

$$\left(\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}^T M^T\right)\left(M\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}\right)=\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}^T\left(M^TM\right)\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}^T\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}=c$$

M viewed as transformation