Vector subspace

Let U be a subset of the underlying set V of a vector space

$$\mathcal{V} = (V, +, \cdot)$$

We say that U is a vector subspace of the vector space $\mathcal{V}$ if

1) $\forall \boldsymbol{x}, \boldsymbol{y} \in U : \boldsymbol{x} + \boldsymbol{y} \in U$

2) $\forall \boldsymbol{x} \in U, \forall a \in R : a \cdot \boldsymbol{x} \in U$

Linear manifold

Let V be a vector space and U its subspace. Further let $\boldsymbol{x}_0 \in V$.

We will call the set $M\left(\boldsymbol{x}_0, U\right) = \left\{\boldsymbol{x}_0 + \boldsymbol{u} \mid \boldsymbol{u} \in U\right\}$ a linear manifold of the vector space V.

We will also write $M = \boldsymbol{x}_0 + U$ or $U = M - \boldsymbol{x}_0$ and say that the manifold M has been created from U through a shift by $\boldsymbol{x}_0$

The dimension of a linear manifold $M\left(U, \boldsymbol{x}_0\right)$ is the dimension of U.

Example

system of equations

$$\begin{array}{ccccccccc} 3x_1 & + & x_2 & - & 4x_3 & + & x_4 & = & -5 \\ x_1 & - & 5x_2 & + & x_3 & - & x_4 & = & 1 \end{array}$$

$$\Downarrow$$

homogenized system

$$\begin{array}{ccccccccc} 3x_1 & + & x_2 & - & 4x_3 & + & x_4 & = & 0 \\ x_1 & - & 5x_2 & + & x_3 & - & x_4 & = & 0 \end{array}$$

$$x_1 = \frac{1}{16}(19s - 4t), x_2 = \frac{1}{16}(7s - 4t), x_3 = s, x_4 = t$$

basis

$$s = 16, t = 0 \Rightarrow x_1 = 19, x_2 = 7, x_3 = 16, x_4 = 0$$

$$s = 0, t = 16 \Rightarrow x_1 = -4, x_2 = -4, x_3 = 0, x_4 = 16$$

$$\boldsymbol{u}_1 = (19, 7, 16, 0)$$

$$\boldsymbol{u}_2 = (-4, -4, 0, 16)$$

All the solutions of

$$\boxed{\begin{array}{ccccccccc} 3x_1 & + & x_2 & - & 4x_3 & + & x_4 & = & 0 \\ x_1 & - & 5x_2 & + & x_3 & - & x_4 & = & 0 \end{array}}$$

form a vector space $U = \mathrm{Lin}\left(\boldsymbol{u}_1, \boldsymbol{u}_2\right)$ with a basis $\boldsymbol{u}_1, \boldsymbol{u}_2$

which is a vector subspace of the vector space

$$V = \left\{\left(x_1, x_2, x_3, x_4\right) \mid x_1, x_2, x_3, x_4 \in R\right\}$$

The original system

$$
\begin{array}{rcrcrcrcr}
3x_1 & + & x_2 & - & 4x_3 & + & x_4 & = & -5 \\
x_1 & - & 5x_2 & + & x_3 & - & x_4 & = & 1
\end{array}
$$

$$\boldsymbol{x}_0 = \left(0, 0, \frac{4}{3}, \frac{1}{3}\right)$$

a particular solution

Thus all the solutions of

$$\begin{array}{rcrcrcrcr} 3x_1 & + & x_2 & - & 4x_3 & + & x_4 & = & -5 \\ x_1 & - & 5x_2 & + & x_3 & - & x_4 & = & 1 \end{array}$$

form a 2-dimensional linear manifold $M = \boldsymbol{x}_0 + \mathrm{Lin}(\boldsymbol{u}_1, \boldsymbol{u}_2)$

$$M = \left\{ (x_1, x_2, x_3, x_4) \;\middle|\; \begin{array}{rcrcrcr} x_1 & = & 0 & + & 19t & - & 4s \\ x_2 & = & 0 & + & 7t & - & 4s \\ x_3 & = & 4/3 & + & 0t & + & 16s \\ x_4 & = & 1/3 & + & 16t & + & 0t \end{array} \right\}$$