

## Vector subspace

Let  $U$  be a subset of the underlying set  $V$  of a vector space

$$V = (V, +, \cdot)$$

We say that  $U$  is a vector subspace of the vector space  $V$  if

1)  $\forall \mathbf{x}, \mathbf{y} \in U : \mathbf{x} + \mathbf{y} \in U$

2)  $\forall \mathbf{x} \in U, \forall a \in R : a \cdot \mathbf{x} \in U$

## Linear manifold

Let  $V$  be a vector space and  $U$  its subspace. Further let  $\mathbf{x}_0 \in V$ .

We will call the set  $M(\mathbf{x}_0, U) = \{\mathbf{x}_0 + \mathbf{u} \mid \mathbf{u} \in U\}$  a linear manifold of the vector space  $V$ .

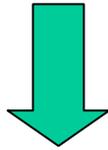
We will also write  $M = \mathbf{x}_0 + U$  or  $U = M - \mathbf{x}_0$  and say that the manifold  $M$  has been created from  $U$  through a shift by  $\mathbf{x}_0$

The dimension of a linear manifold  $M(U, \mathbf{x}_0)$  is the dimension of  $U$ .

## Example

$$3x_1 + x_2 - 4x_3 + x_4 = -5$$

$$x_1 - 5x_2 + x_3 - x_4 = 1$$



$$3x_1 + x_2 - 4x_3 + x_4 = 0$$

$$x_1 - 5x_2 + x_3 - x_4 = 0$$

$$x_1 = \frac{1}{16}(19s - 4t), x_2 = \frac{1}{16}(7s - 4t), x_3 = s, x_4 = t$$

$$s = 16, t = 0 \Rightarrow x_1 = 19, x_2 = 7, x_3 = 16, x_4 = 0$$

$$s = 0, t = 16 \Rightarrow x_1 = -4, x_2 = -4, x_3 = 0, x_4 = 16$$

system of equations

homogenized system

basis

$$\mathbf{u}_1 = (19, 7, 16, 0)$$

$$\mathbf{u}_2 = (-4, -4, 0, 16)$$

All the solutions of

$$\begin{array}{rcccccc} 3x_1 & + & x_2 & - & 4x_3 & + & x_4 & = & 0 \\ x_1 & - & 5x_2 & + & x_3 & - & x_4 & = & 0 \end{array}$$

form a vector space  $U = \text{Lin}(\mathbf{u}_1, \mathbf{u}_2)$  with a basis  $\mathbf{u}_1, \mathbf{u}_2$

which is a vector subspace of the vector space

$$V = \{(x_1, x_2, x_3, x_4) \mid x_1, x_2, x_3, x_4 \in R\}$$

## The original system

$$\begin{array}{rclclcl} 3x_1 & + & x_2 & - & 4x_3 & + & x_4 & = & -5 \\ x_1 & - & 5x_2 & + & x_3 & - & x_4 & = & 1 \end{array}$$

$$\mathbf{x}_0 = \left( 0, 0, \frac{4}{3}, \frac{1}{3} \right)$$

a particular solution

Thus all the solutions of

$$\begin{aligned} 3x_1 + x_2 - 4x_3 + x_4 &= -5 \\ x_1 - 5x_2 + x_3 - x_4 &= 1 \end{aligned}$$

form a 2-dimensional linear manifold  $M = \mathbf{x}_0 + \text{Lin}(\mathbf{u}_1, \mathbf{u}_2)$

$$M = \left\{ (x_1, x_2, x_3, x_4) \left| \begin{array}{l} x_1 = 0 + 19t - 4s \\ x_2 = 0 + 7t - 4s \\ x_3 = 4/3 + 0t + 16s \\ x_4 = 1/3 + 16t + 0t \end{array} \right. \right\}$$