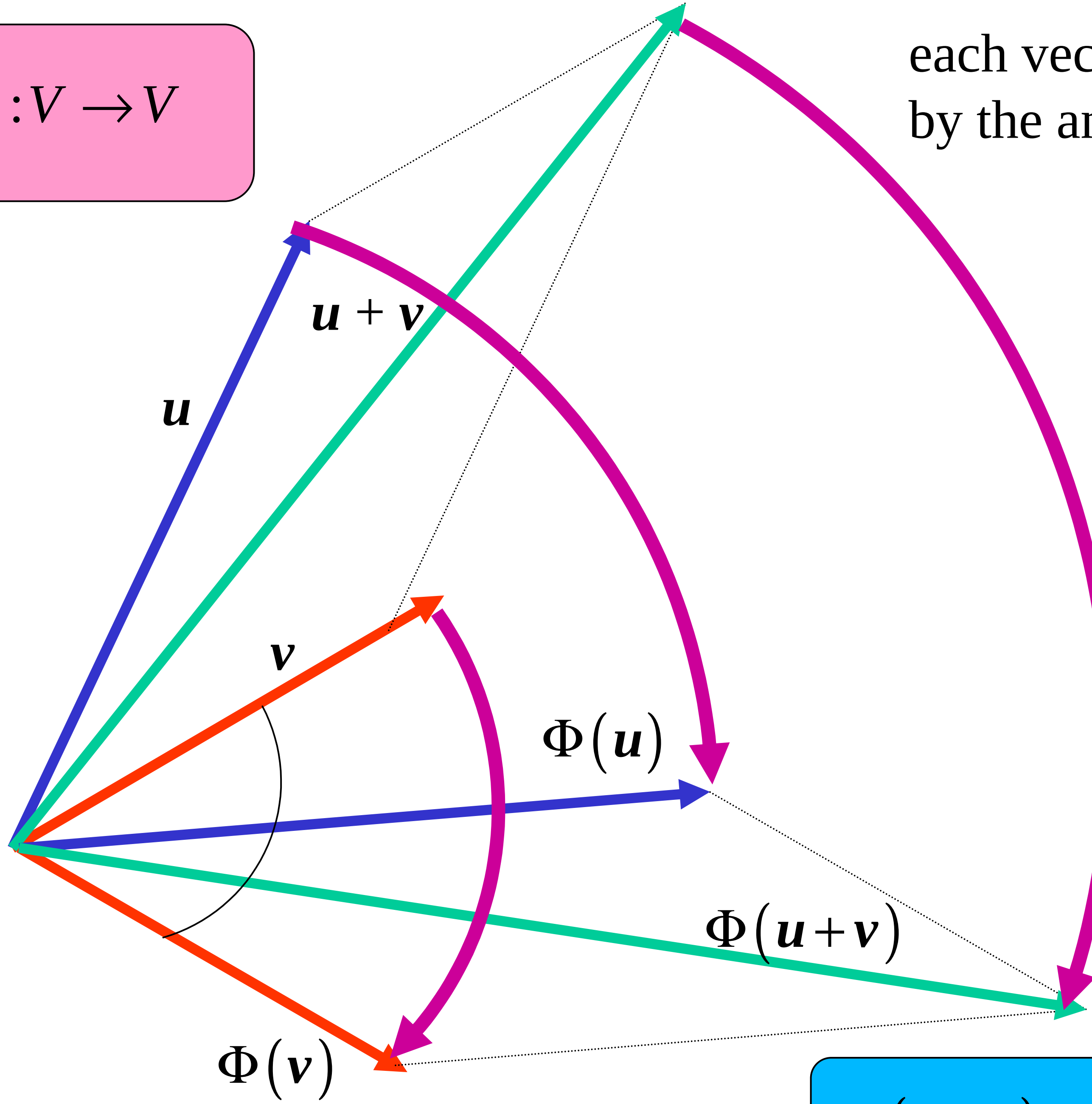
$$\Phi : V \to V$$

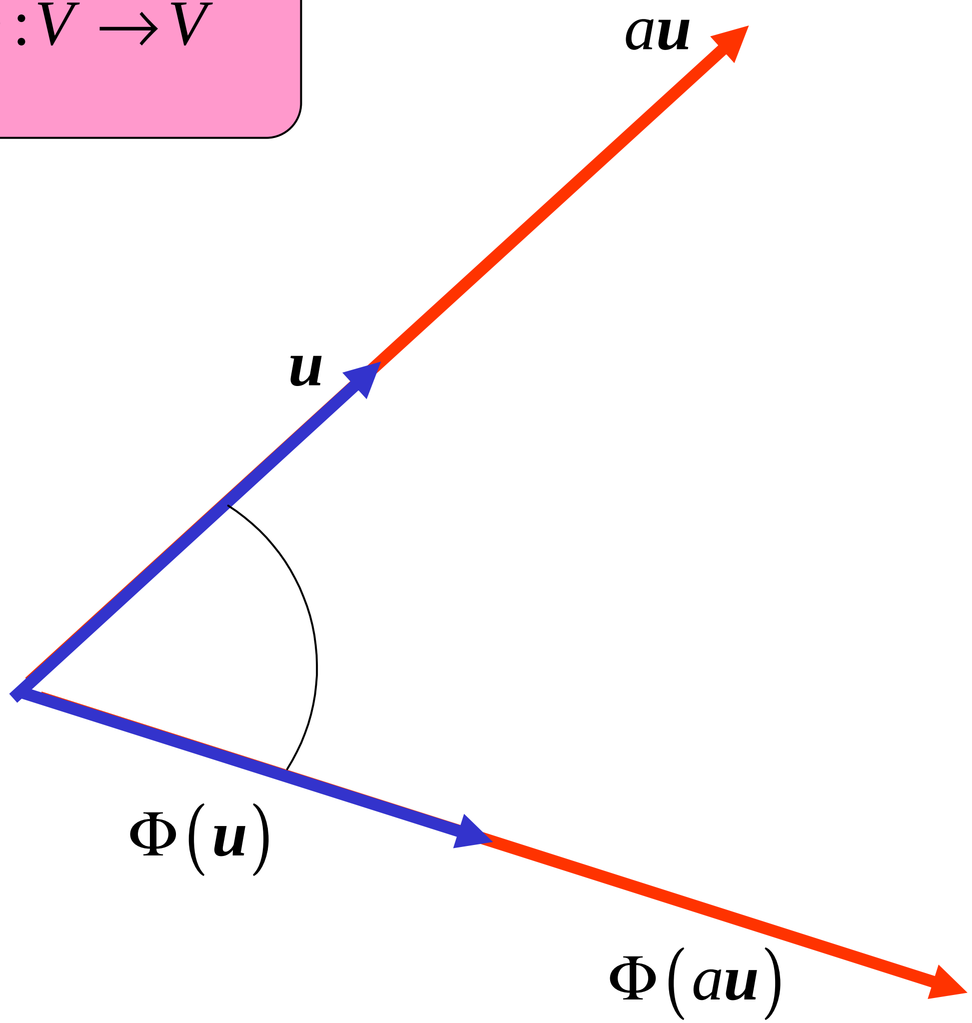
each vector is rotated by the angle φ



$$\Phi(\boldsymbol{u}+\boldsymbol{v}) = \Phi(\boldsymbol{u}) + \Phi(\boldsymbol{v})$$

$$\Phi : V \to V$$

each vector extended by a



$$\Phi(a\boldsymbol{u}) = a\Phi(\boldsymbol{u})$$

Another example

For an $a \in R$, consider the mapping $a : V \to V$ defined by

$$a(\boldsymbol{u}) = a \cdot \boldsymbol{u}$$

each vector is extended by *a*

We have

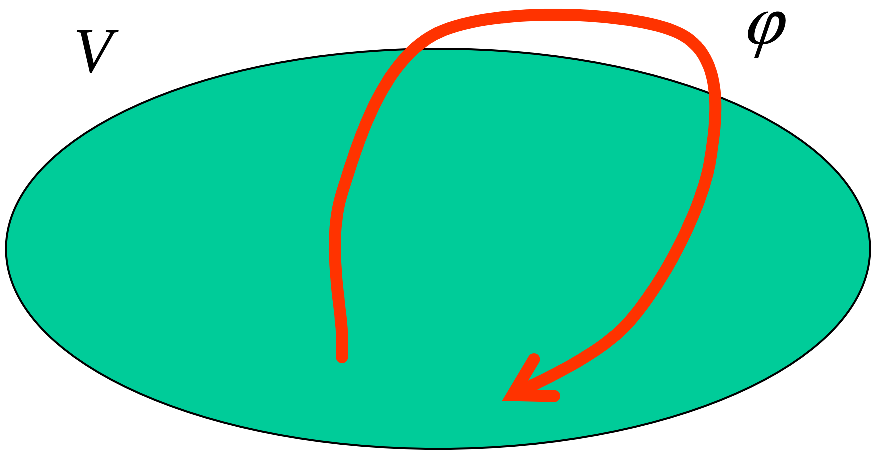
$$\underline{a(\boldsymbol{u}+\boldsymbol{v})} = a \cdot (\boldsymbol{u}+\boldsymbol{v}) = a \cdot \boldsymbol{u} + a \cdot \boldsymbol{v} = \underline{a(\boldsymbol{u}) + a(\boldsymbol{v})}$$

$$\underline{a(b \cdot \boldsymbol{u})} = a \cdot (b \cdot \boldsymbol{u}) = ab \cdot \boldsymbol{u} = ba \cdot \boldsymbol{u} = b \cdot (a \cdot \boldsymbol{u}) = \underline{b \cdot a(\boldsymbol{u})}$$

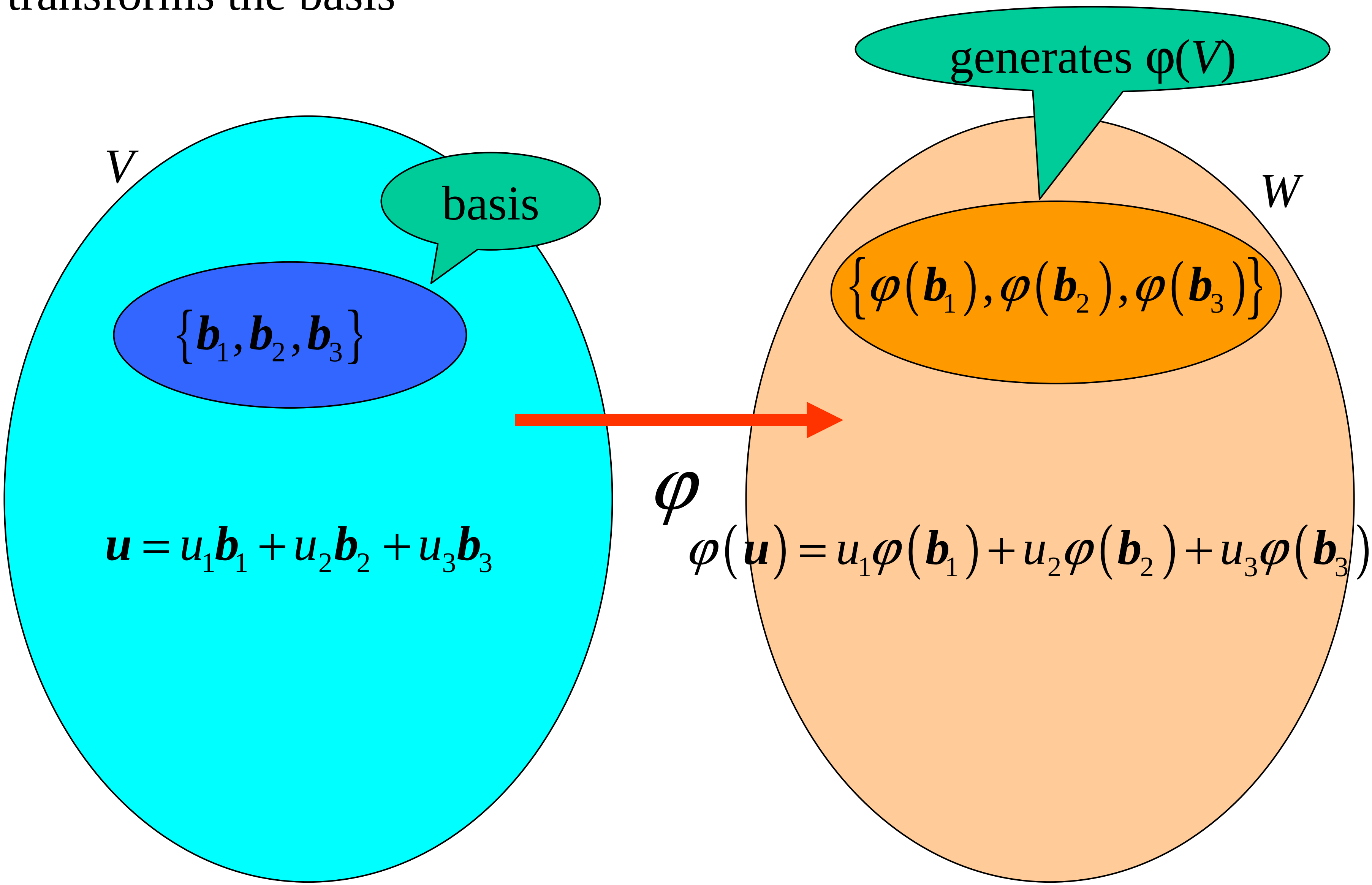
Let V and W be vector spaces. A mapping $\varphi : V \to W$ is called a linear transformation of V into W if the following axioms are satisfied:

- $\varphi(\boldsymbol{u}+\boldsymbol{v})=\varphi(\boldsymbol{u})+\varphi(\boldsymbol{v})$
- $\varphi(a\cdot\boldsymbol{u})=a\cdot\varphi(\boldsymbol{u})$

- φ preserves linear combinations of vectors.
- φ also maps the zero vector of *V* to the zero vector of *W*.
- a linear transformation that maps a vector space into itself is sometimes called a **<u>linear operator</u>**.



To define a linear transformation it is sufficient to show how it transforms the basis



Let us, for example, consider a linear operator φ on V_3:

basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$

$$\varphi(\boldsymbol{b}_1) = t_{11}\boldsymbol{b}_1 + t_{12}\boldsymbol{b}_2 + t_{13}\boldsymbol{b}_3$$

$$\varphi(\boldsymbol{b}_2) = t_{21}\boldsymbol{b}_1 + t_{22}\boldsymbol{b}_2 + t_{23}\boldsymbol{b}_3$$

$$\varphi(\boldsymbol{b}_3) = t_{31}\boldsymbol{b}_1 + t_{32}\boldsymbol{b}_2 + t_{33}\boldsymbol{b}_3$$

$$\boldsymbol{u} = u_1\boldsymbol{b}_1 + u_2\boldsymbol{b}_2 + u_3\boldsymbol{b}_3 \qquad \varphi(\boldsymbol{u}) = u_1'\boldsymbol{b}_1 + u_2'\boldsymbol{b}_2 + u_3'\boldsymbol{b}_3$$

$$\begin{aligned}
\varphi(\boldsymbol{u}) &= u_1(t_{11}\boldsymbol{b}_1 + t_{12}\boldsymbol{b}_2 + t_{13}\boldsymbol{b}_3) + \\
&\quad + u_2(t_{21}\boldsymbol{b}_1 + t_{22}\boldsymbol{b}_2 + t_{23}\boldsymbol{b}_3) + \\
&\quad + u_3(t_{31}\boldsymbol{b}_1 + t_{32}\boldsymbol{b}_2 + t_{33}\boldsymbol{b}_3) = \\
&= (u_1t_{11} + u_2t_{21} + u_3t_{31})\boldsymbol{b}_1 + \\
&\quad (u_1t_{12} + u_2t_{22} + u_3t_{32})\boldsymbol{b}_2 + \\
&\quad (u_1t_{13} + u_2t_{23} + u_3t_{33})\boldsymbol{b}_3
\end{aligned}$$

$$\begin{array}{cc} & \begin{array}{ccc} \varphi(\boldsymbol{b}_1) & \varphi(\boldsymbol{b}_1) & \varphi(\boldsymbol{b}_1) \end{array} \\ T = & \begin{pmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{pmatrix} \end{array}$$

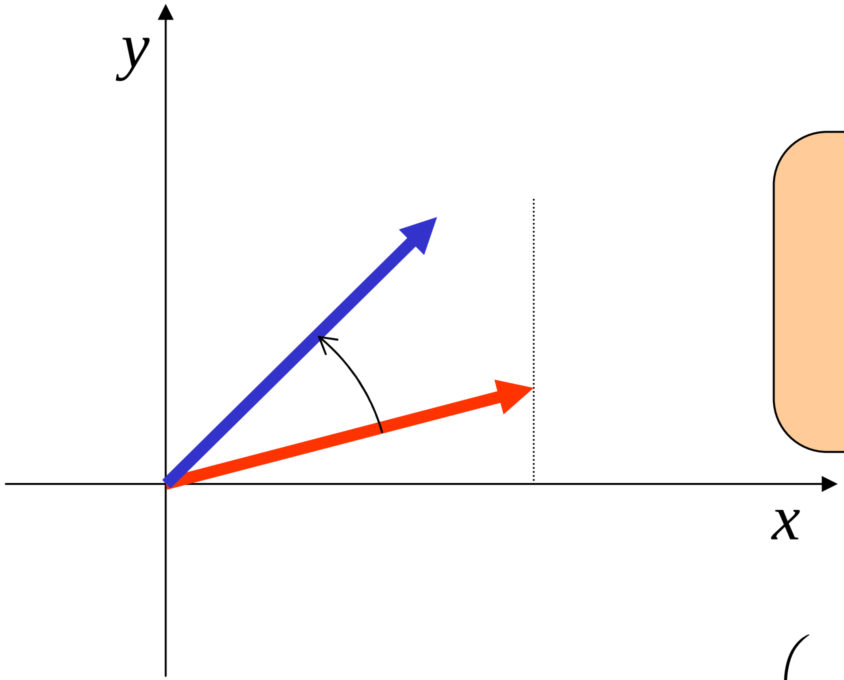
If we write the coordinates of vectors as one-column matrices, the previous example transformation may be written as follows:

$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Each m by n matrix represents a linear transformation of an m-dimensional vector space V into an n-dimensional vector space W.

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} t_{11} & t_{21} & \cdots & t_{m1} \\ t_{12} & t_{22} & \cdots & t_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{1n} & t_{2n} & \cdots & t_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}$$

Example



$$\Phi : V_2 \rightarrow V_2$$

each vector is rotated by φ = 30° anti-clockwise

$$T = \begin{pmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$$