Example

Consider a linear operator in V_3 given by the following matrix:

$$T = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

The vector (1,2,1) is transformed into the vector (2,4,2):

$$\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$

which has the same effect as the transformation

$$2 \cdot (1,2,1) = (2,4,2)$$

Given a linear operator represented by a square matrix, how can we find the vectors, if any, that this operator just extends by λ?

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

$$A\boldsymbol{x} = \lambda \boldsymbol{x}$$

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\left(A-\lambda I\right) = \boldsymbol{o} \qquad (*)$$

The system (*) is homogeneous and, as such, has only a non-trivial solution when

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$$\Downarrow$$

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n = 0$$

characteristic polynomial of A

The real roots $\lambda_1, \lambda_2, \ldots, \lambda_r, 0 \le r \le n$ of

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n = 0$$

are called the **eigenvalues** of the matrix A and, for each λ_i

$$\begin{pmatrix} a_{11}-\lambda_i & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda_i & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda_i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a non-trivial solution, which is called the **eigenvector** of A corresponding to the eigenvalue λ_i

Example

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \qquad \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

$\boxed{\lambda_1 = 1}$

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 0 \\ -x_1 & + & x_2 & - & x_3 & = & 0 \\ x_1 & - & x_2 & + & x_3 & = & 0 \end{array}$$

$$x_1 = -t, x_2 = 0, x_3 = t$$

Eigenvector $(-t, 0, t)$

$\boxed{\lambda_1 = 2}$

$$\begin{array}{rcrcrcl} & & x_2 & + & x_3 & = & 0 \\ -x_1 & & & - & x_3 & = & 0 \\ x_1 & - & x_2 & & & = & 0 \end{array}$$

$$x_1 = -t, x_2 = -t, x_3 = t$$

Eigenvector $(-t, -t, t)$

$\boxed{\lambda_1 = 3}$

$$\begin{array}{rcrcrcl} -x_1 & + & x_2 & + & x_3 & = & 0 \\ -x_1 & - & x_2 & - & x_3 & = & 0 \\ x_1 & - & x_2 & - & x_3 & = & 0 \end{array}$$

$$x_1 = 0, x_2 = -t, x_3 = t$$

Eigenvector $(0, -t, t)$

Example

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ -4 & 4-\lambda & 0 \\ -2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0 \qquad \lambda_1, \lambda_2, \lambda_3 = 2$$

$$\begin{aligned} -2x_1 &+ x_2 &= 0 \\ -4x_1 &+ 2x_2 &= 0 \\ -2x_1 &+ x_2 &= 0 \end{aligned}$$

$$\Downarrow$$

$$-2x_1 + x_2 = 0$$

$$\Downarrow$$

$$x_1 = \frac{1}{2}s \qquad x_2 = s \qquad x_3 = t$$

In this case, the eigenvectors form a two dimensional subspace generated, for example, by the vectors

$$\boldsymbol{u}_1 = (1, 2, 0), \boldsymbol{u}_2 = (0, 0, 1)$$