

Example

Consider a linear operator in V_3 given by the following matrix:

$$T = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

The vector $(1,2,1)$ is transformed into the vector $(2,4,2)$:

$$\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$

which has the same effect as the transformation

$$2 \cdot (1, 2, 1) = (2, 4, 2)$$

Given a linear operator represented by a square matrix, how can we find the vectors, if any, that this operator just extends by λ ?

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

$$Ax = \lambda x$$

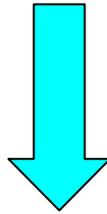
$$\begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(A - \lambda I) = \mathbf{o}$$

(*)

The system (*) is homogeneous and, as such, has only a non-trivial solution when

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$



$$\lambda^n + c_1 \lambda^{n-1} + \dots + c_{n-1} \lambda + c_n = 0$$

characteristic polynomial of A

The real roots $\lambda_1, \lambda_2, \dots, \lambda_r, 0 \leq r \leq n$ of

$$\lambda^n + c_1 \lambda^{n-1} + \dots + c_{n-1} \lambda + c_n = 0$$

are called the **eigenvalues** of the matrix A and, for each λ_i

$$\begin{pmatrix} a_{11} - \lambda_i & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda_i & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda_i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a non-trivial solution, which is called the **eigenvector** of A corresponding to the eigenvalue λ_i

Example

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2 - \lambda & 1 & 1 \\ -1 & 2 - \lambda & -1 \\ 1 & -1 & 2 - \lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

$$\lambda_1 = 1$$

$$x_1 + x_2 + x_3 = 0$$

$$-x_1 + x_2 - x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$x_1 = -t, x_2 = 0, x_3 = t$$

Eigenvector $(-t, 0, t)$

$$\lambda_1 = 2$$

$$x_2 + x_3 = 0$$

$$-x_1 - x_3 = 0$$

$$x_1 - x_2 = 0$$

$$x_1 = -t, x_2 = -t, x_3 = t$$

Eigenvector $(-t, -t, t)$

$$\lambda_1 = 3$$

$$-x_1 + x_2 + x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

$$x_1 - x_2 - x_3 = 0$$

$$x_1 = 0, x_2 = -t, x_3 = t$$

Eigenvector $(0, -t, t)$

Example

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ -4 & 4 - \lambda & 0 \\ -2 & 1 & 2 - \lambda \end{vmatrix} = 0$$

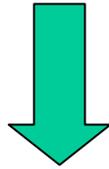
$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

$$\lambda_1, \lambda_2, \lambda_3 = 2$$

$$-2x_1 + x_2 = 0$$

$$-4x_1 + 2x_2 = 0$$

$$-2x_1 + x_2 = 0$$



$$-2x_1 + x_2 = 0$$



$$x_1 = \frac{1}{2}s \quad x_2 = s \quad x_3 = t$$

In this case, the eigenvectors form a two dimensional subspace generated, for example, by the vectors

$$\mathbf{u}_1 = (1, 2, 0), \mathbf{u}_2 = (0, 0, 1)$$