<u>Function</u>

Let $f : A \ \mathbf{a} \ B$ be a mapping. If both A and B are sets of real numbers, we say that f is a function or, more precisely, a real function of one variable.

When defining a function f of x sometimes denoted as $f(x)$ or $y = f(x)$, we start with determining its domain then defining the functional procedure itself. Sometimes the first step is omitted and then the function domain is determined as the maximum set of values of x for which the function value $f(x)$ can be obtained.

Example

$$f(x)=\frac{\sqrt{x^2-1}}{x}$$

Clearly, if we want to use the above formula to determine the function value of f at x, the expression under the square root sign must not be negative and the value of the fraction denominator must be different from zero.

This leaves us with a set $A = D(f)$ that can be expressed as

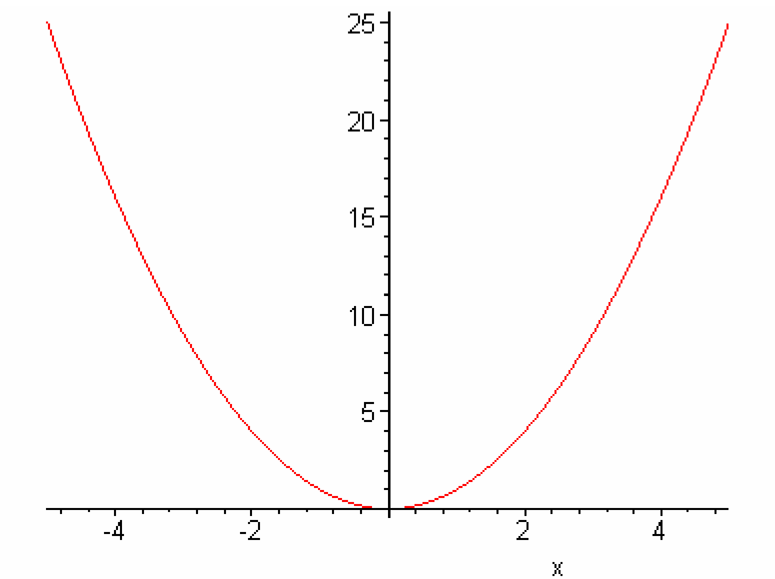
$$D(f)=(-\infty;-1]\cup[1;\infty)$$

Graph of a function

A function $y = f(x)$ may be represented geometrically by its *graph*.

When plotting the graph of a function $y = f(x)$, we consider each value that the variable x may assume to be the x-coordinate and y the y-coordinate of a point on a plotting plane.

For $y = x^2$, the result then may look something like this:



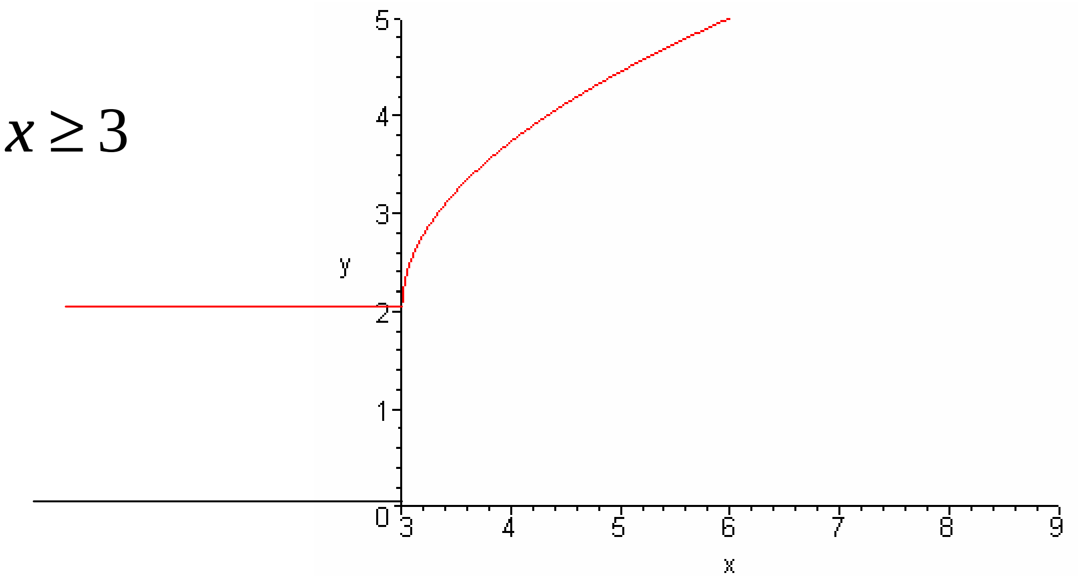
Other ways of defining a function

Most often, the rules used to determine the function values are presented in the form of a simple formula to be calculated by hand such as $y = 3x^2 - 5x + 7$ or using a pocket calculator such as

$$y = \sqrt{\ln x - x}$$

Sometimes we resort to more or less intuitive "instructions for use" such as

$$f(x) = \begin{cases} \sqrt{3x - 9} + 2 \text{ for } x \geq 3 \\ 2 \text{ otherwise} \end{cases}$$



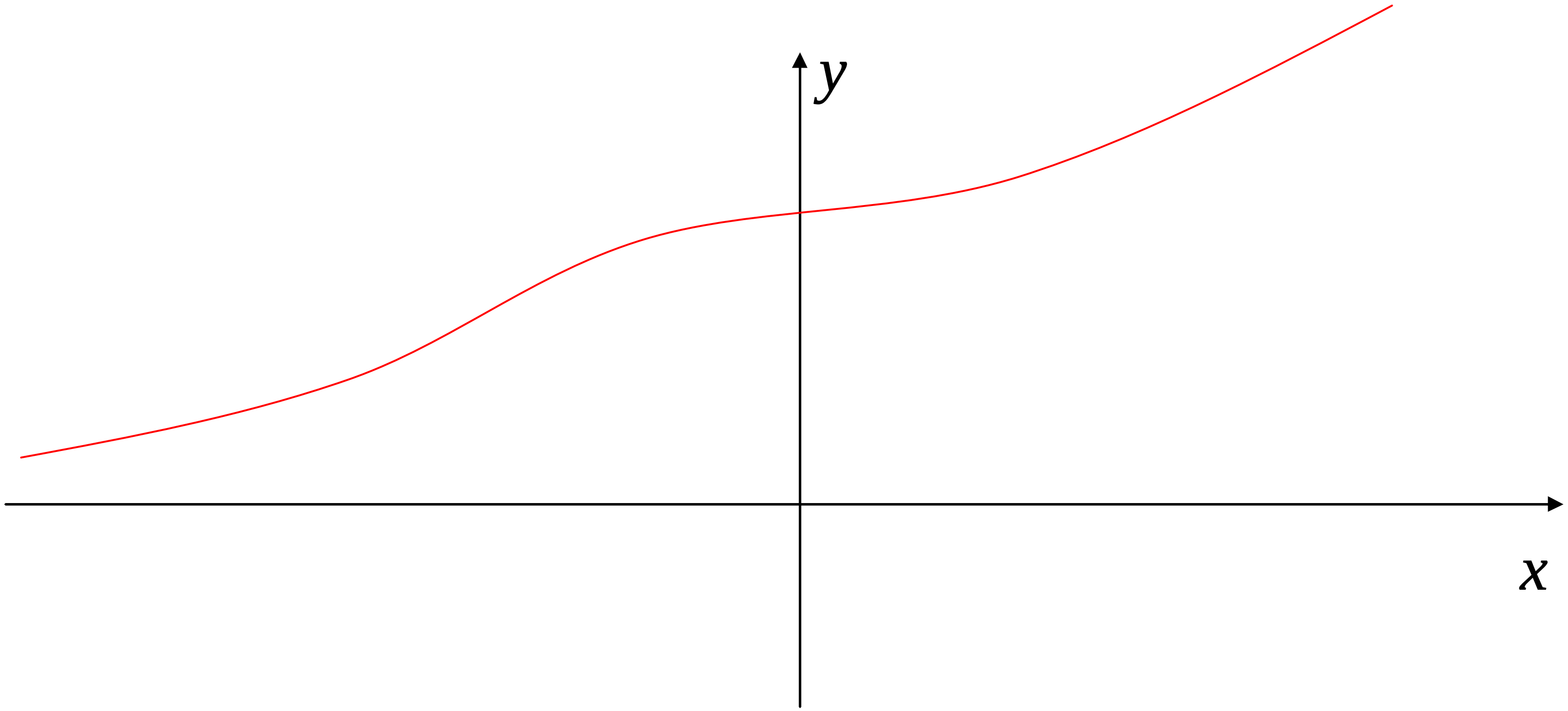
Other ways of defining a function

One way to define a function, which was used mostly in the pre-pocket-calculator era was using tables of pre-calculated values such as for logarithms and trigonometric functions.

If the function is a sophisticated one, we may even have to use a programming language to write a computer program calculating its values.

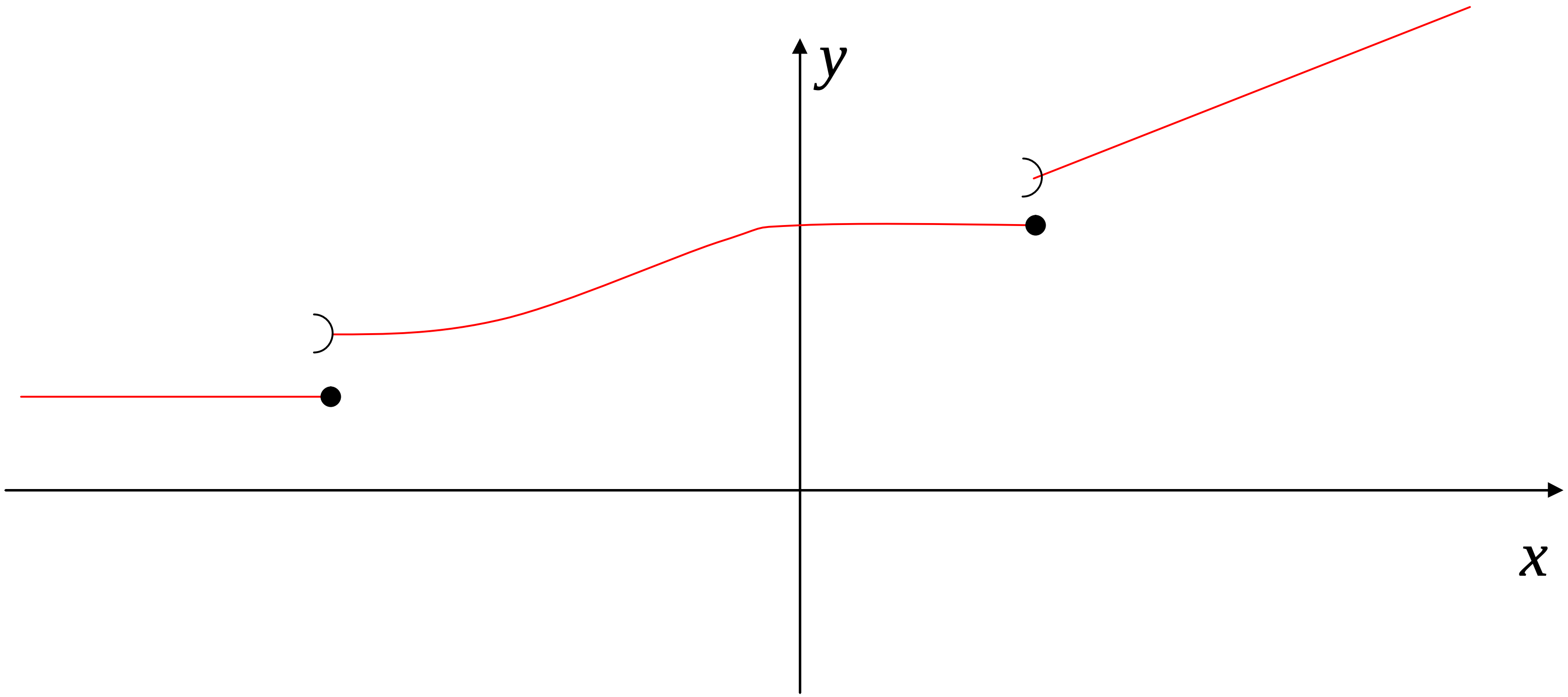
Increasing function

We say that a function $y = f(x)$ is increasing if, for every $x_1, x_2 \in D(f)$, $x_1 < x_2$, we have $f(x_1) < f(x_2)$



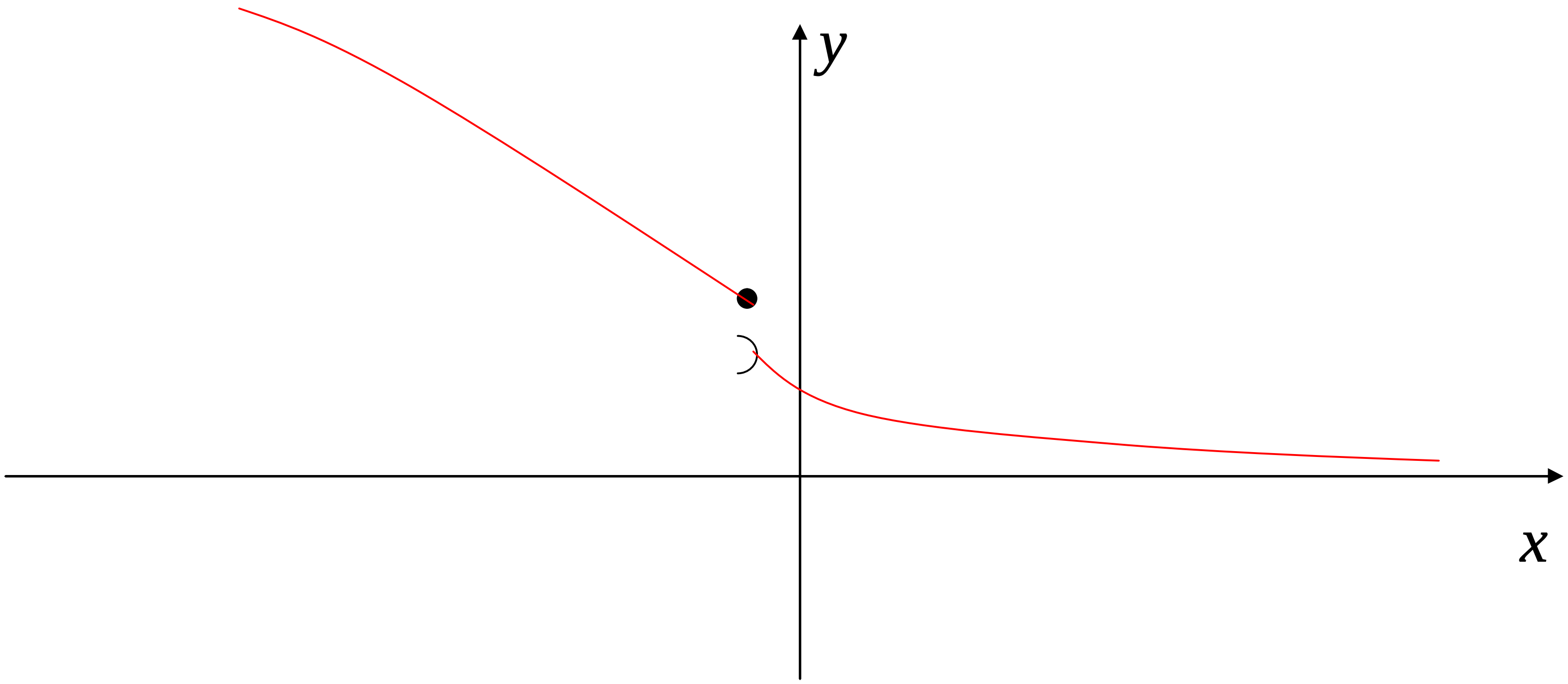
Non-decreasing function

We say that a function $y = f(x)$ is non-decreasing if, for every $x_1, x_2 \in D(f)$, $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$



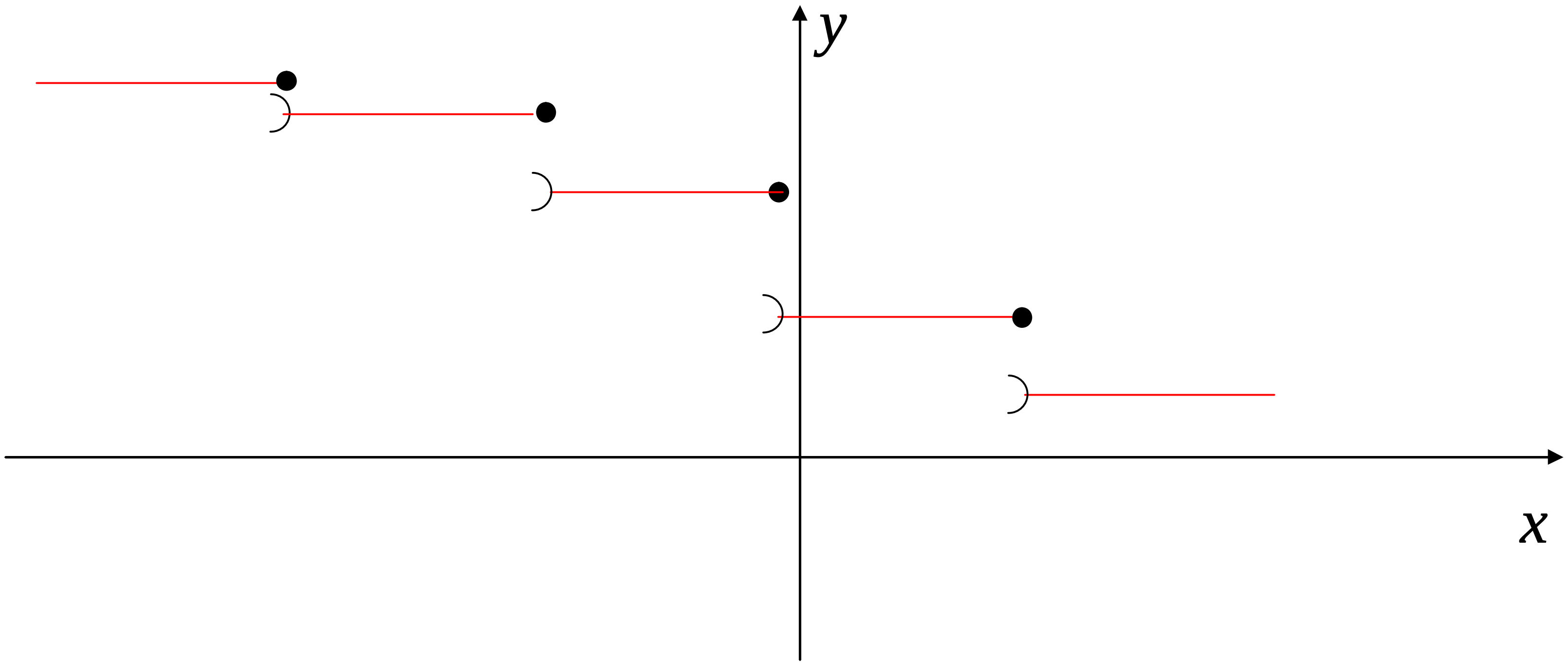
Decreasing function

We say that a function $y = f(x)$ is decreasing if, for every $x_1, x_2 \in D(f)$, $x_1 < x_2$, we have $f(x_1) > f(x_2)$



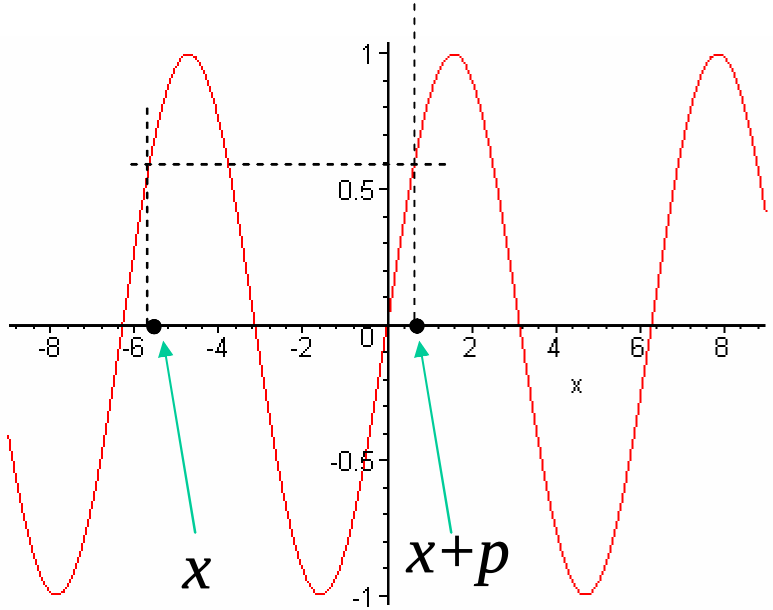
Non-increasing function

We say that a function $y = f(x)$ is non-increasing if, for every $x_1, x_2 \in D(f)$, $x_1 < x_2$, we have $f(x_1) \geq f(x_2)$



Periodic function

We say that a function $y = f(x)$ is periodic with period $p > 0$ if, for every $x \in D(f)$, $f(x+p)$ exists and $f(x+p) = f(x)$.



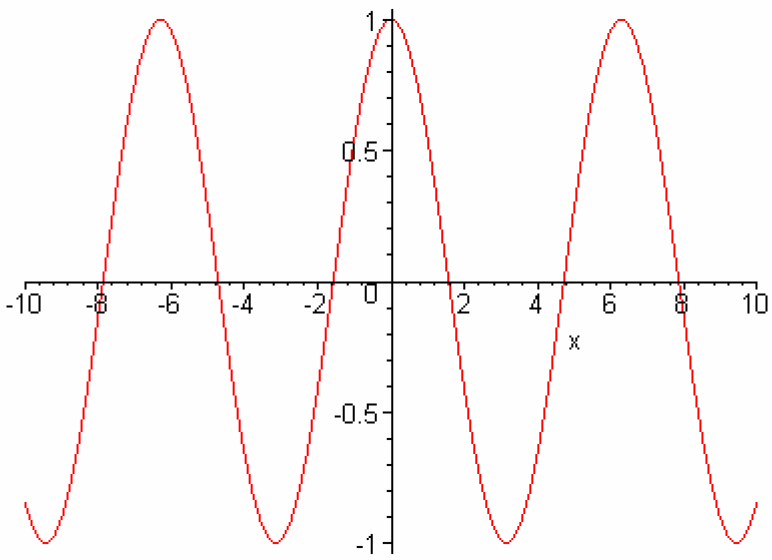
Clearly, if a function is periodic with period *p*, it is also periodic with a period equal to any multiple *kp* of *p*. We usually use the smallest of such periods.

Even function

A function $f(x)$ is even if, for every $x \in D(f)$, we have $-x \in D(f)$ and

$$f(-x) = f(x)$$

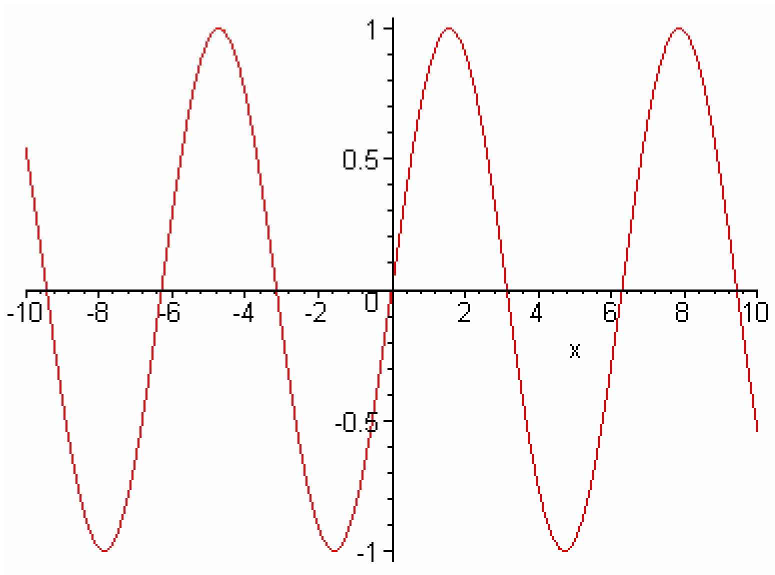
$$f(-x) = f(x)$$



Odd function

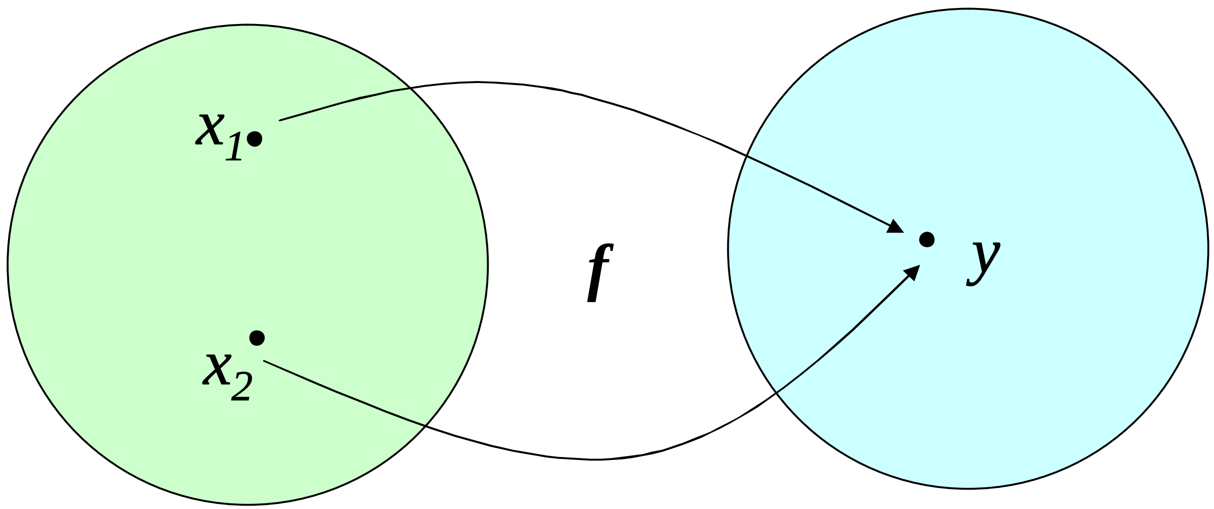
A function $f(x)$ is odd if, for every $x \in D(f)$, we have $-x \in D(f)$ and

$$f(-x) = -f(x)$$



Inverse function

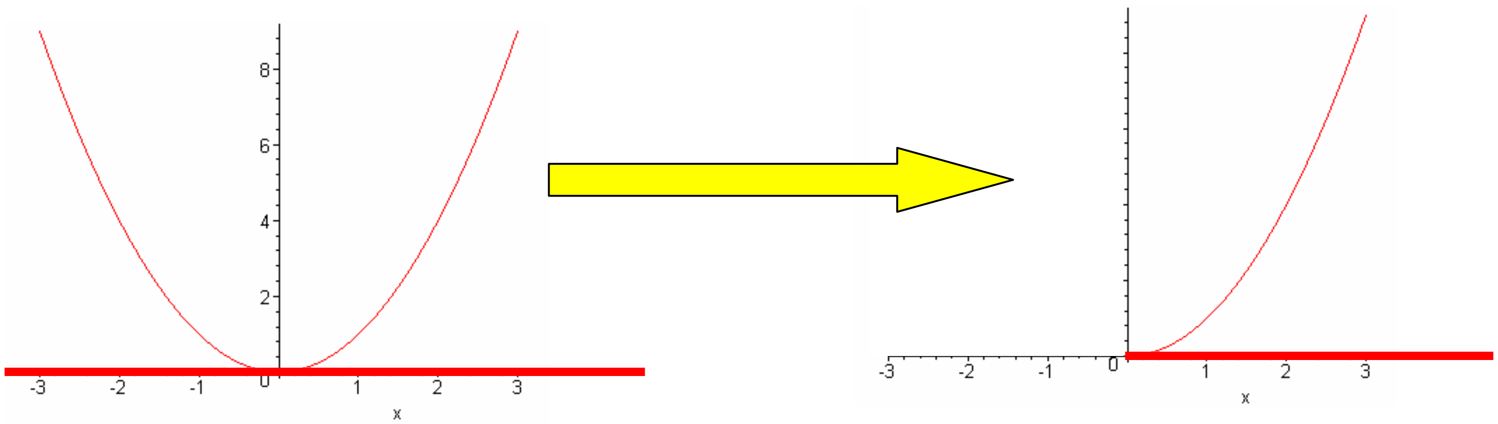
Let $y = f(x)$ be a function. Then we could define a new function $x = g(y)$: to every y we assign x such that $y = f(x)$. In other words, we swap the elements in each pair of the function viewed as a relation. However, as the figure below shows, we cannot do this with every function.



We cannot unambiguously assign a single element to y as a function would require.

We have to restrict ourselves to functions that are one-to-one mappings. This can also be done by choosing a suitable subset of the domain.

Example: The function $y = x^2 : R \;\mathbf{a}\; R^+$ clearly has no inverse since, for example, $(-1)^2 = 1^2$. However, if we take the function $y = x^2 : R^+ \;\mathbf{a}\; R^+$ instead, the problem is solved. (Here R^+ denotes the set of non-negative real numbers).



If $y = f(x)$ is a function to which the inverse function exists, we denote it by $x = f^{-1}(y)$ or sometimes we write $y = f^{-1}(x)$.

The graph of the inverse function $f^{-1}(x)$ to a function $f(x)$ can be drawn as a mirror image of the graph of $f(x)$ with respect to the graph of the function $y = x$. Note that the function domain and range change places.

$y = f(x)$ $\qquad\qquad$ $y = f^{-1}(x)$