Polynomial function

The function $f : R \to R$ defined as

$$y = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

where $a_0, a_1, \ldots, a_n$, $a_0 \neq 0$ are real numbers and $n \geq 0$ is called a ***polynomial function*** of degree n.

The numbers $a_0, a_1, \ldots, a_n$, a_0 are its *coefficients* and a_0 is its *leading coefficient*.

The right-hand-side expression itself is called a *real polynomial* of degree n.

Often, we are interested in finding a number r such that, for a polynomial function $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$

we have $f(r) = 0$. Such a number is called a ***root*** of the polynomial

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

In such a situation, we write

$$\boxed{a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0}$$

and say that we are looking for a solution or for the solutions to the *polynomial equation*.

Not every polynomial has a root as the example of a quadratic polynomial, that is, a polynomial of degree two, shows.

The roots of the quadratic equation $ax^2 + bx + c = 0$ can be calculated by the formula

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$$

where $D = b^2 - 4ac$ is the ***discriminant*** of the equation.

If D < 0, no real root exists.

Let a real polynomial $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ be given.

The following properties of $f(x)$ can be proved:

- $f(x)$ has no more than n roots

- any polynomial $f(x)$ of an odd degree has at least one real root

- if r is a root of $f(x)$, then we can write $f(x) = (x - r)\, g(x)$ where $g(x)$ is a polynomial of degree $n - 1$

For every real polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

we can write

$$f(x) = a_0 [p_1(x)]^{k_1} \cdots [p_r(x)]^{k_r} [q_1(x)]^{m_1} \cdots [q_s(x)]^{m_s}$$

$$p_i(x) = x^2 + b_i x + c_i \text{ with } b_i^2 - 4c_i < 0$$

$$q_i(x) = x - r_i$$

$f(x)$ then has the roots $r_1, r_2, \ldots, r_s$ with multiplicities $m_1, m_2, \ldots, m_s$

Let $f(x) = x^n + a_1 x^{n-1} + \mathbf{L} + a_{n-1} x + a_n$ be a real polynomial with integer coefficients a_1, ..., a_n.

If r is an integer root of $f(x)$, then r is a divisor of a_n.

Example

Let us try to find a root r of $x^3 - 4x^2 - 26x + 35$

Since r has to be a divisor of 35, the only candidates are

1, -1, 5, -5, 7, -7, 35, -35. By substituting each of these, we will eventually find that only 7 is a root.

Horner's method

We may use this method when calculating the value of a polynomial $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ at r.

We have

$$f(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n$$

and we can also write

$$f(r) = (x - r)\left(b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-2} x + b_{n-1}\right) + b_n$$

From the last equation it follows that $f(r) = b_n$

Performing the multiplication on the right-hand side and comparing the coefficients yields the following formulas:

$$b_0 = a_0, \quad b_{i+1} = r b_i + a_{i+1}, \quad i = 1, 2, \ldots n - 1$$

For practical calculations we can use the following two-row table

	a_0	a_1	a_2	$\cdots$	a_{n-1}	a_n
r	b_0	b_1	b_2	$\cdots$	b_{n-1}	b_n

After entering the coefficients a_0, a_1, ..., a_n in the first row and r in the leftmost lower field, we start filling in the second row by first copying a_0 into the lower field next to r and then keep multiplying the number in lower field i by the number in the leftmost lower field, adding the number from the upper field i+1 and placing the result in the lower field i+1 until we reach the lower field n.

Example

Find the value of $x^6 - 5x^5 + 3x^3 - x + 10$ at 3.

	1	-5	0	3	0	-1	10
3	1	-2	-6	-15	-45	-136	-398

$$3 \cdot 1 - 5 = -2$$

$$3 \cdot (-2) - 0 = -6$$

$$3 \cdot (-6) + 3 = -15$$

$$3 \cdot (-15) + 0 = -45$$

$$3 \cdot (-45) - 1 = -136$$

$$3 \cdot (-136) + 10 = -398$$

$$x^6 - 5x^5 + 3x^3 - x + 10 = (x-3)\left(x^5 - 2x^4 - 6x^3 - 15x^2 - 45x - 136\right) - 398$$