Limit

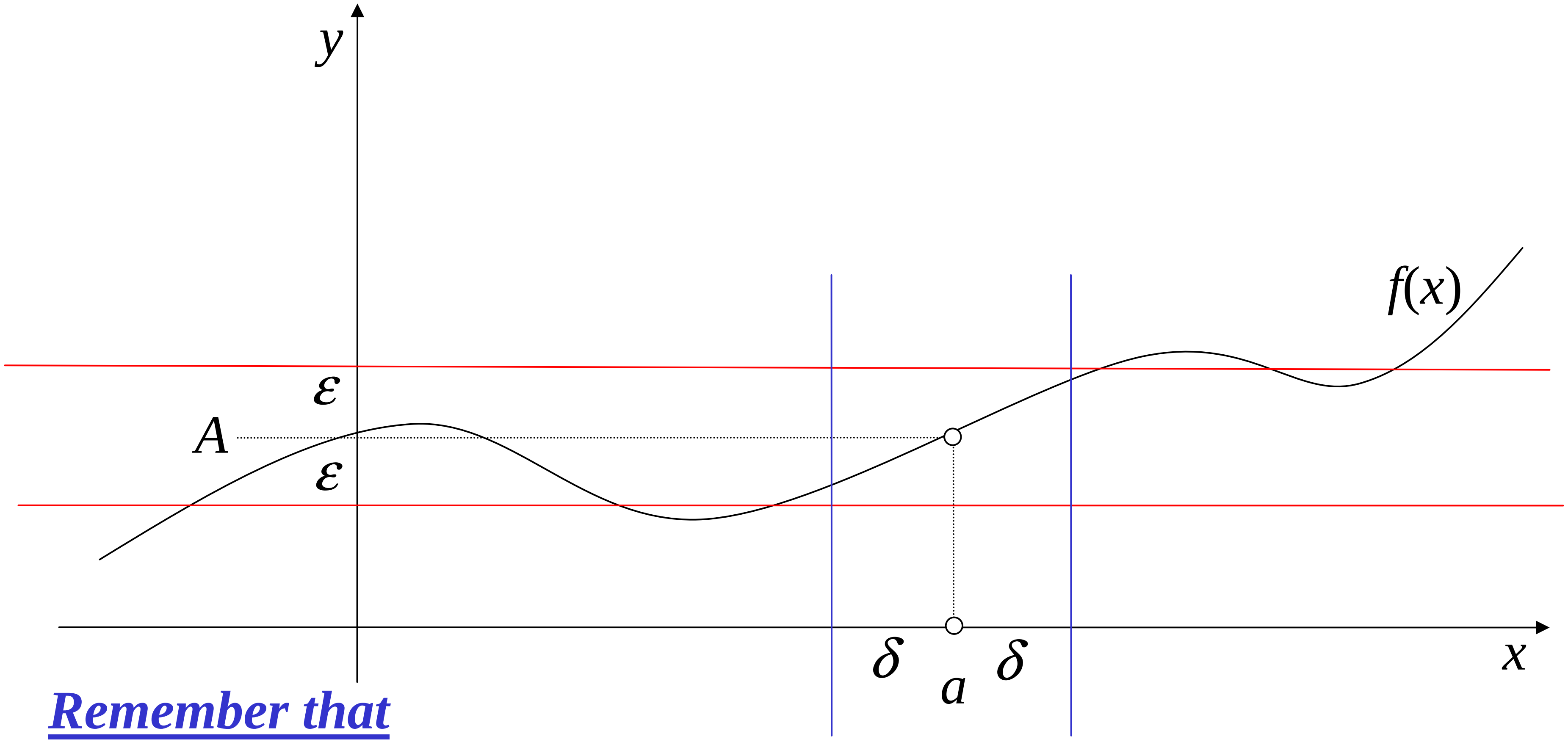
We say that a function $f(x)$ has a limit A at a point a, formally

$$\lim_{x\to a} f(x) = A$$

if, for every $\varepsilon > 0$, we can find $\delta > 0$ such that

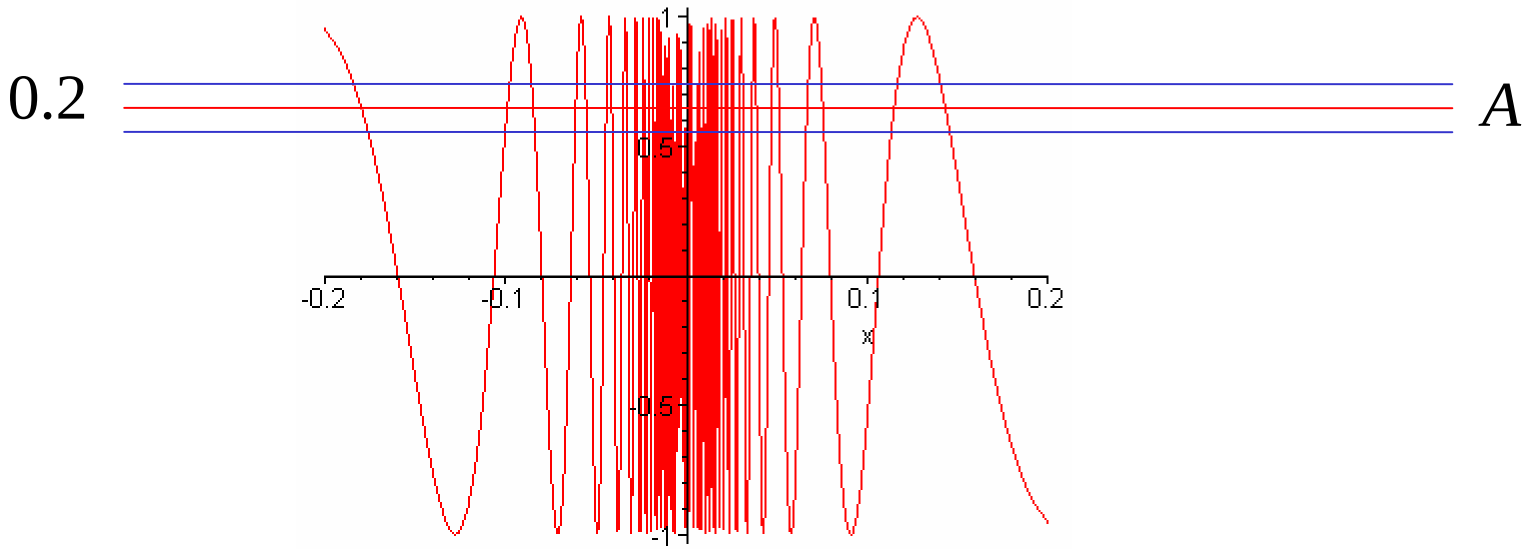
$$\left|A - f(x)\right| < \varepsilon$$

for every $x \in (a-\delta, a) \cup (a, a+\delta)$



Remember that

- we are actually not interested in what happens with $f(x)$ at a, but rather what goes on in each, however small, neighbourhood of a
- important is that $\varepsilon > 0$ is introduced arbitrarily as a first idea, but $\delta > 0$ has to be then found accordingly to define all the values of x for which the condition $|A - f(x)| < \varepsilon$ is satisfied



No limit of $\sin(1/x)$ exists for x approaching 0. Indeed, for any $A \in (-1,1)$ we can take, say, $\varepsilon = 0.1$ and then it is easy to see that for any $\delta > 0$, we can find $x_0 \in (-\delta, 0) \cup (0, \delta)$ such that either $\sin(x_0) > A + 0.1$ or $\sin(x_0) < A - 0.1$

Limit from the left

We say that a function $f(x)$ has a limit A from the left or on the left at a point a, formally $\lim_{x \to a-} f(x) = A$

if, for every $\varepsilon > 0$, we can find $\delta > 0$ such that $|A - f(x)| < \varepsilon$

for every $x \in (a - \delta, a)$

Limit from the right

We say that a function $f(x)$ has a limit A from the right or on the right at a point a, formally $\lim\limits_{x \to a-} f(x) = A$

if, for every $\varepsilon > 0$, we can find $\delta > 0$ such that $|A - f(x)| < \varepsilon$

for every $(a, a + \delta)$

A function $f(x)$ has a limit at a point a if it has a limit from the left at a and a limit from the right at a and if these limits are the same.

$$\lim_{x \to a} f(x) = A \Leftrightarrow \lim_{x \to a-} f\left(x\right) = A \wedge \lim_{x \to a+} f\left(x\right) = A$$

Limit for *x* tending to infinity

We say that a function $f(x)$ has a limit A for x tending to infinity

$$\lim_{x \to \infty} f(x) = A$$

if, for every $\varepsilon > 0$, we can find x_0 such that

$$\left| A - f(x) \right| < \varepsilon$$

for every $x > x_0$

Limit for *x* tending to minus infinity

We say that a function $f(x)$ has a limit A for x tending to minus infinity

$$\lim_{x \to -\infty} f(x) = A$$

if, for every $\varepsilon > 0$, we can find x_0 such that

$$\left| A - f(x) \right| < \varepsilon$$

for every $x < x_0$

Improper limit

We say that a function $f(x)$ has the limit ∞ at a point a, formally

$$\lim_{x \to a} f(x) = \infty$$

if, for every K, we can find $\delta > 0$ such that

$$f(x) > K$$

for every $x \in (a - \delta, a) \cup (a, a + \delta)$

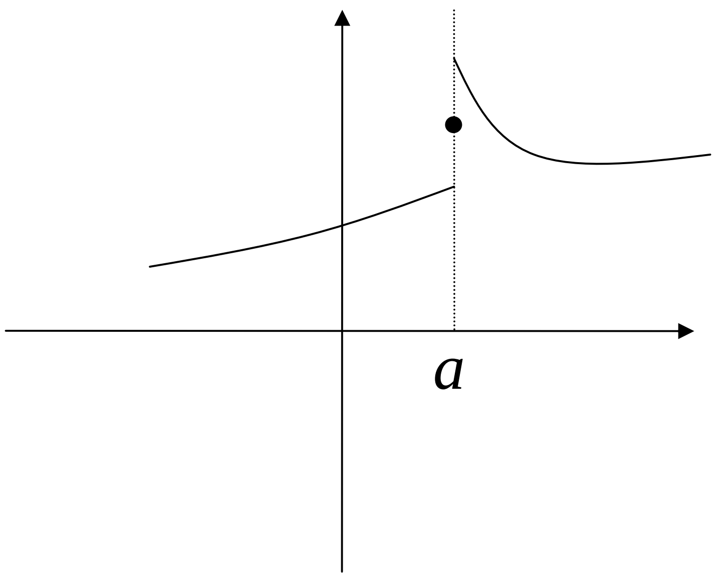
Improper limit at $-\infty$ is defined in an analogous way.

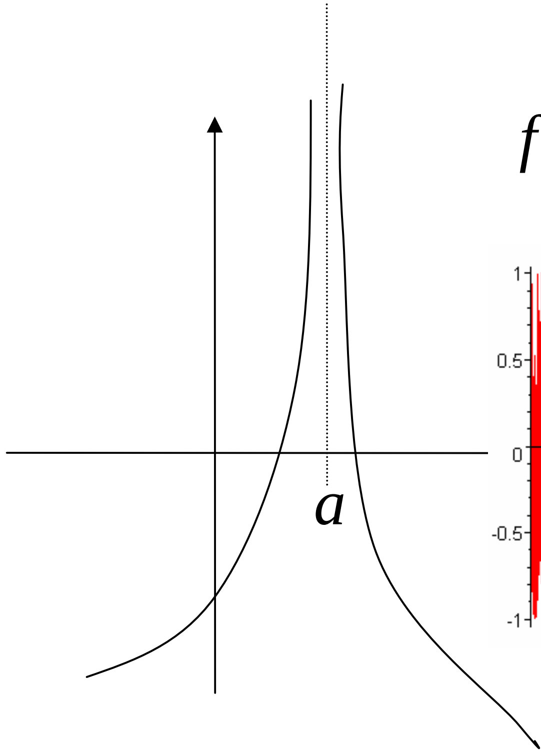
A function has at most one limit at a given point

<u>**Continuous function**</u>

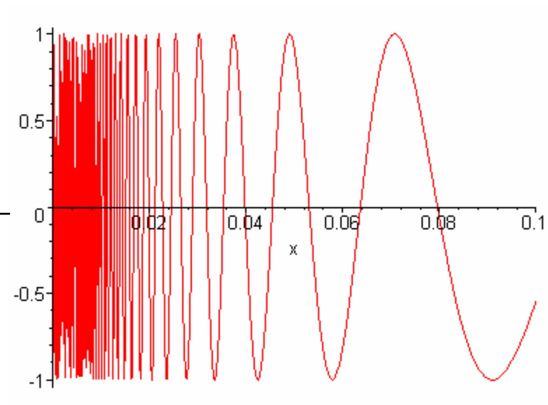
We say that a function $f(x)$ is continuous at a if it has a limit

$\lim_{x \to a} f(x) = A$, is defined at a and $f(a) = A$





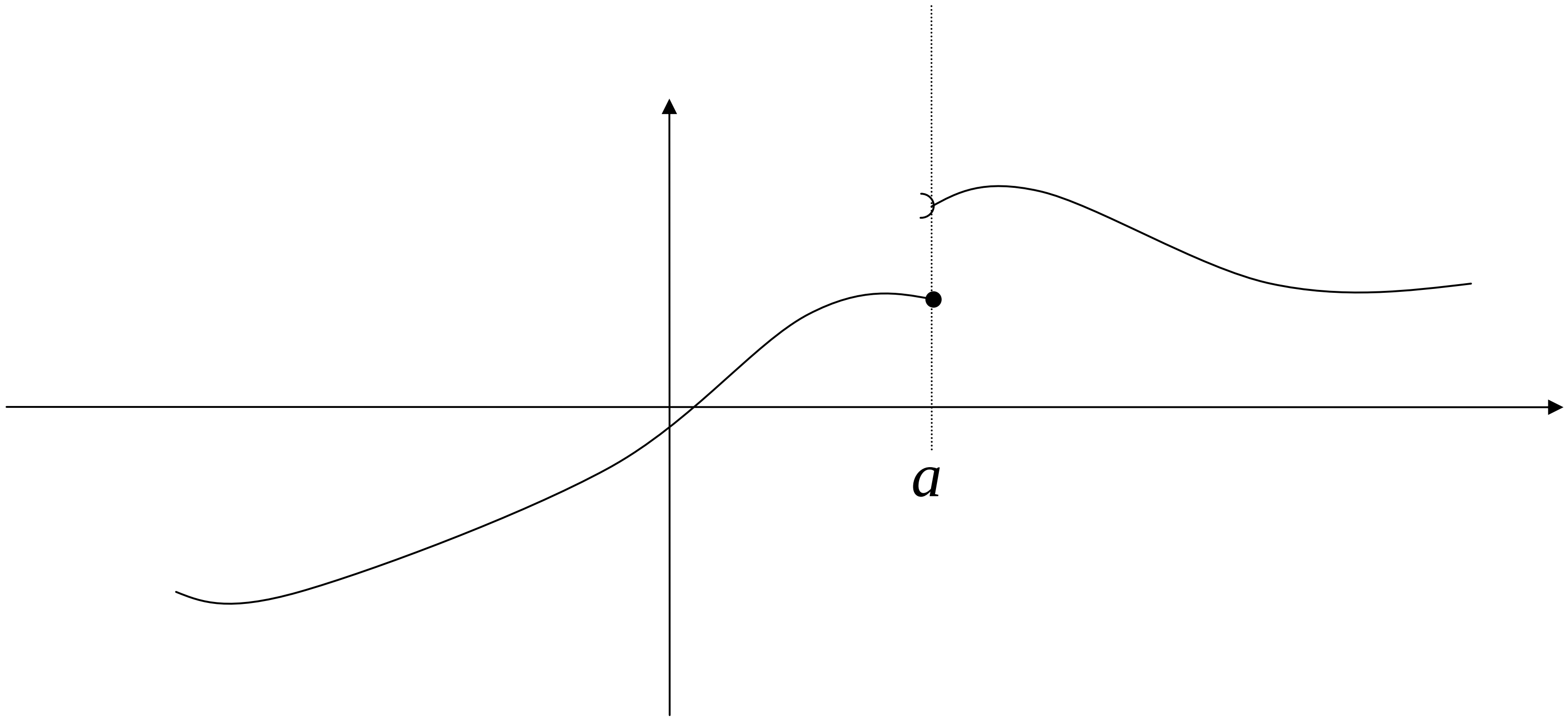
$$f(x) = \sin\left(\frac{1}{x}\right)$$



at $x = 0$

Examples of non-continuous functions

Continuous functions on the left or on the right are defined in much the same way using the concepts of limit on the left or limit on the right



The above function is continuous on the left at *a* but not on the right.

Examples

$$\lim_{x\to 1} x^3 - 5x^2 + 2 = 1^3 - (5)1^2 + 2 = -2$$

$$\lim_{x\to \pi/4} \sin x = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\lim_{x\to 2} \frac{1+\cos^2 x}{\ln(x+1)} = \frac{1+\cos^2 2}{\ln 3} = 1.819369805\ldots$$

If it causes no problems, that is, if the function is defined at the point *a* to which *x* approaches and is continuous at it, we can just substitute *a* for *x* and calculate the value of the function at *a*.

Example

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)\left(x^2+x+1\right)}{(x-1)}=\lim_{x\to 1}\left(x^2+x+1\right)=3$$

Since we are not interested in the point 1 itself but only in any of its $(1-a,1)\cup(1,1+a)$ neighbourhoods, we can cancel the expressions $(x-1)$. The resulting function then causes no more problems.

Some other easy limits

- $\lim_{x \to a} b = b$ the limit of a constant function
- $\lim_{x \to \infty} \frac{a}{x} = \lim_{x \to -\infty} \frac{a}{x} = 0$ for $a \neq 0$
- $\lim_{x \to 0+} \frac{a}{x} = \infty, \quad \lim_{x \to 0-} \frac{a}{x} = -\infty,$ for $a > 0$
- $\lim_{x \to \infty} e^x = \infty, \quad \lim_{x \to \infty} \ln x = \infty, \quad \lim_{x \to -\infty} e^x = 0, \quad \lim_{x \to 0} \ln x = -\infty$
- $\lim_{x \to \infty} \arctan x = \frac{\pi}{2} \quad \lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$

Some useful rules:

- $$\lim_{x\to a}\left(f(x)\pm g(x)\right)=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$$

- $$\lim_{x\to a}\left(f(x)\cdot g(x)\right)=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$$

- $$\lim_{x\to a}\left(f(x)/g(x)\right)=\lim_{x\to a} f(x)/\lim_{x\to a} g(x)\quad \text{if } \lim_{\mathrm{x}\to \mathrm{a}} g(x)\neq 0$$

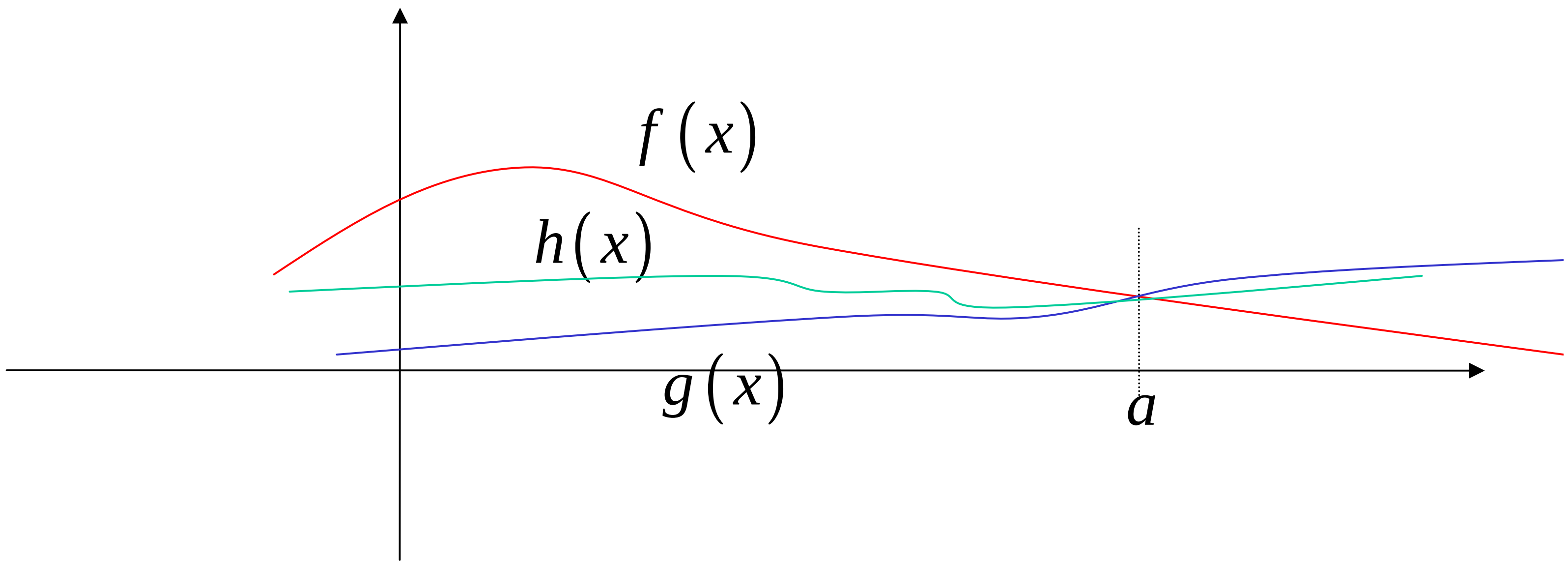
- $$\lim_{x\to a}\left|f(x)\right|=\left|\lim_{x\to a} f(x)\right|$$

always provided that both limits on the right-hand side exist *"squeezing rule"*

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = A$ and

$h(x)$ lies between $f(x)$ and $g(x)$ for $x \in (a-\delta, a) \cup (a, a+\delta)$

then $\lim_{x \to a} h(x) = A$



Limit of a composite function

Let $\lim_{x\to a} g(x) = b$, and $\lim_{u\to b} f(u) = c$

Let a number $h > 0$ exists such that $g(x) \neq b$ for every $x \in (a-h, a) \cup (a, a+h)$

Then $\lim_{x\to a} f(g(x)) = c$

Example

$$\lim_{x\to 0+} \arctan\frac{1}{x} = \frac{\pi}{2}$$

Since $\lim_{x\to 0+} \frac{1}{x} = \infty$ and $\lim_{u\to\infty} \arctan u = \frac{\pi}{2}$

we have

$$\lim_{x\to\infty} \arctan\frac{1}{x} = \frac{\pi}{2}$$

Example

$$\lim_{x\to\infty}\frac{2x^2-8x+12}{-x^2-10}=\lim_{x\to\infty}\frac{\frac{2x^2}{x^2}-\frac{8x}{x^2}+\frac{12}{x^2}}{-\frac{x^2}{x^2}-\frac{10}{x^2}}=\lim_{x\to\infty}\frac{2-\frac{8}{x}+\frac{12}{x^2}}{-1-\frac{10}{x^2}}=\frac{2-0-0}{-1-0}=-2$$

Using the same method, it is not difficult to prove that

- $$\lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)}=\infty \text{ if } n>m$$

- $$\lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)}=0 \text{ if } n<m$$

where P_n and Q_m are polynomials of degree n and m respectively and

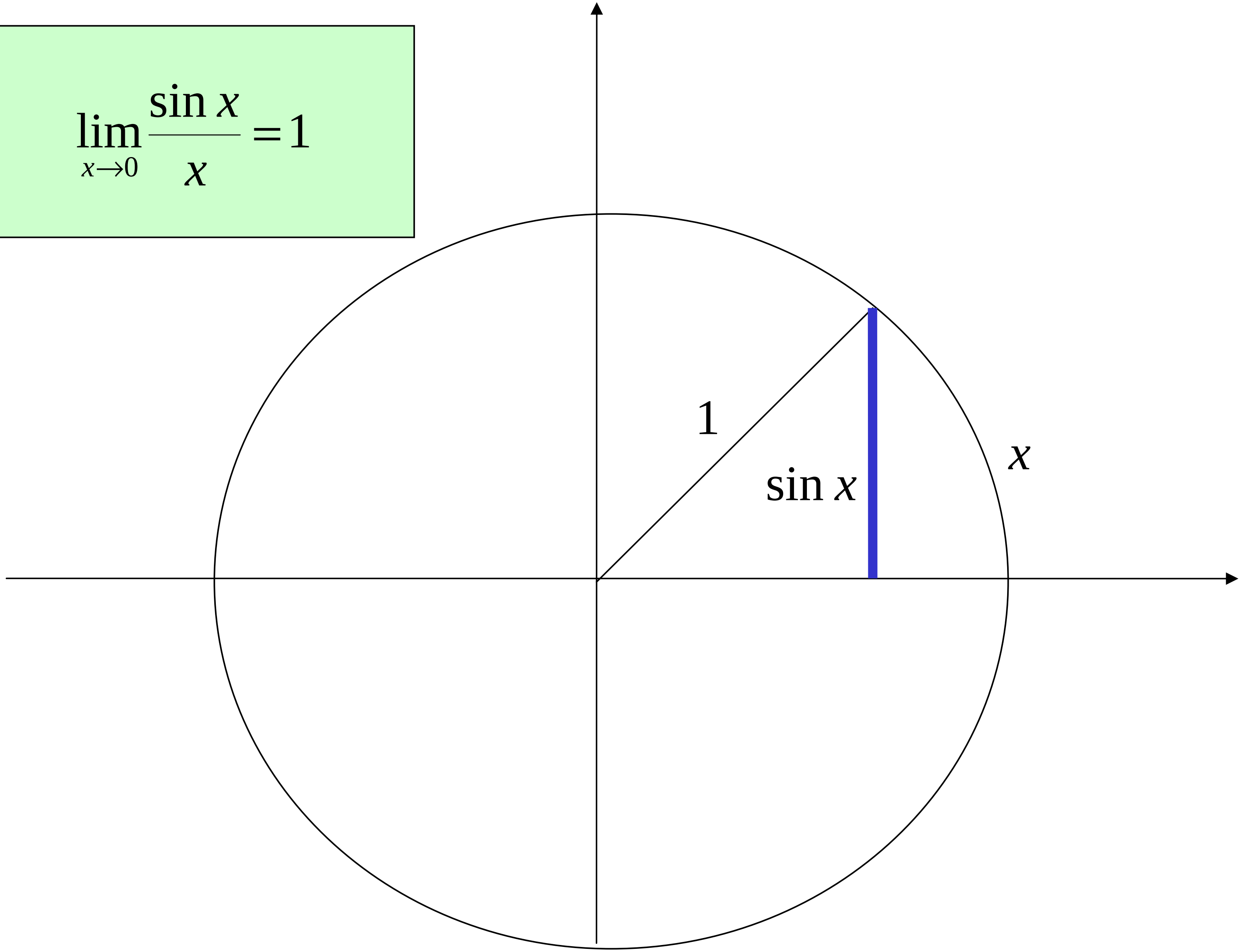
- $$\lim_{x\to\infty}\frac{a_n x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}{b_n x^n+b_{n-1}x^{n-1}+\cdots+b_1x+b_0}=\frac{a_n}{b_n}$$

$$\lim_{x\to\infty}\frac{e^x}{P_n(x)}=\infty,\quad \lim_{x\to\infty}\frac{P_n(x)}{e^x}=0 \text{ where } P_n(x) \text{ is a polynomial}$$

$$\lim_{x\to\infty}\frac{\ln x}{P_n(x)}=0,\quad \lim_{x\to\infty}\frac{P_n(x)}{\ln x}=\infty \text{ where } P_n(x) \text{ is a polynomial}$$

As x tends to infinity, e^x grows much faster than any polynomial and $\ln x$ grows much slower than any polynomial.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



$$\boxed{\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a}$$

and as a special case we have

$$\boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = 1}$$