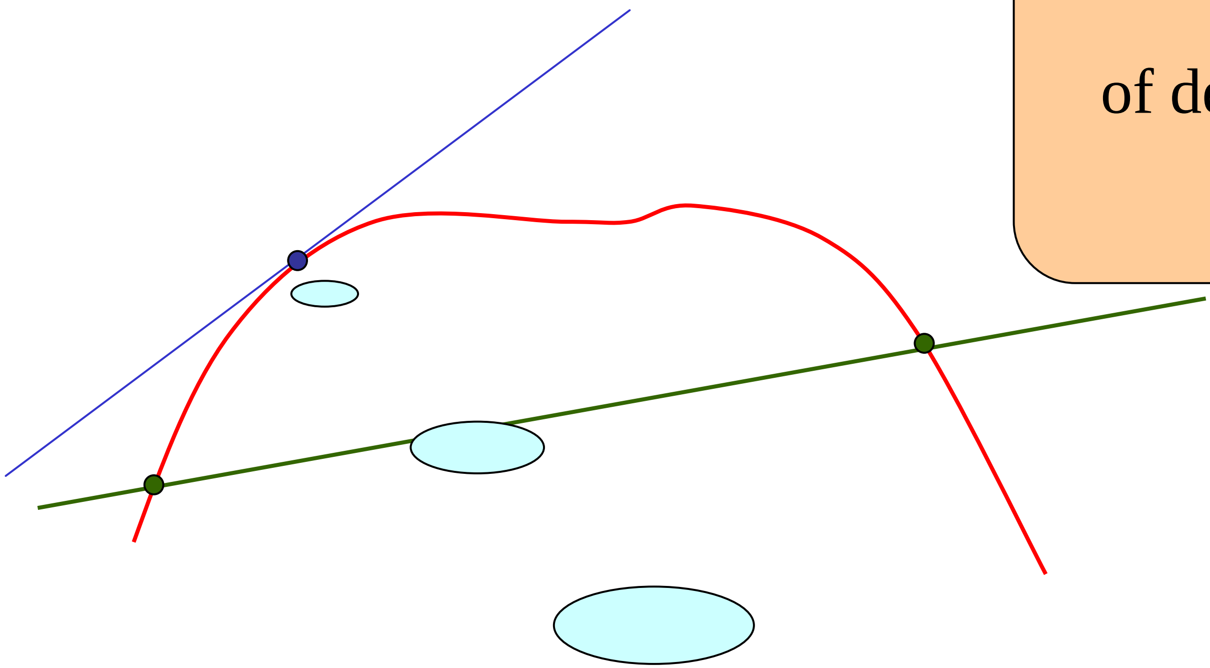
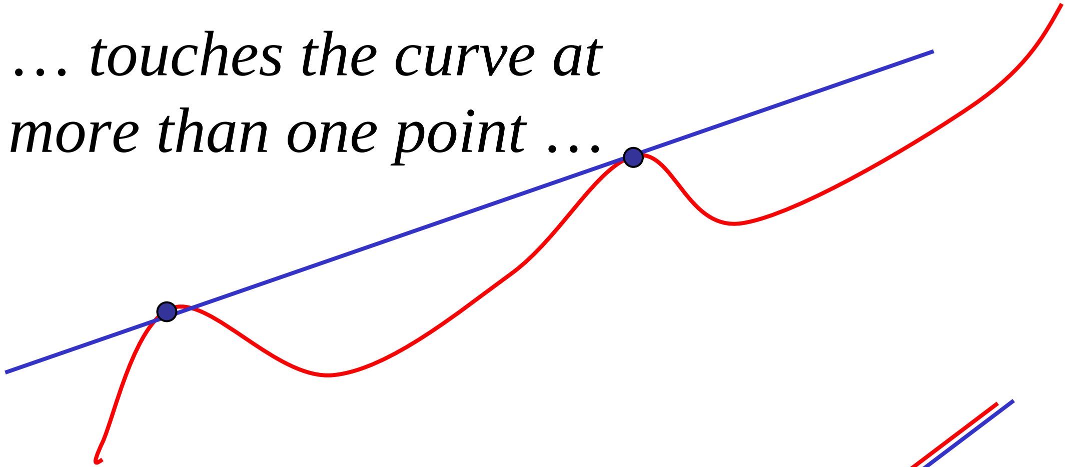
What is the best way of defining a tangent line to a curve?

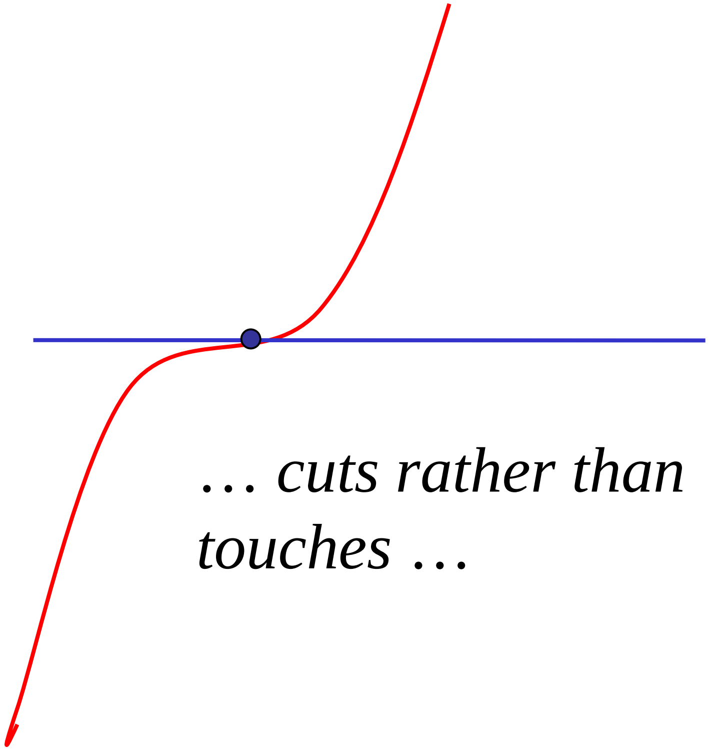


... a tangent line to a curve touches it at a single point ... ?

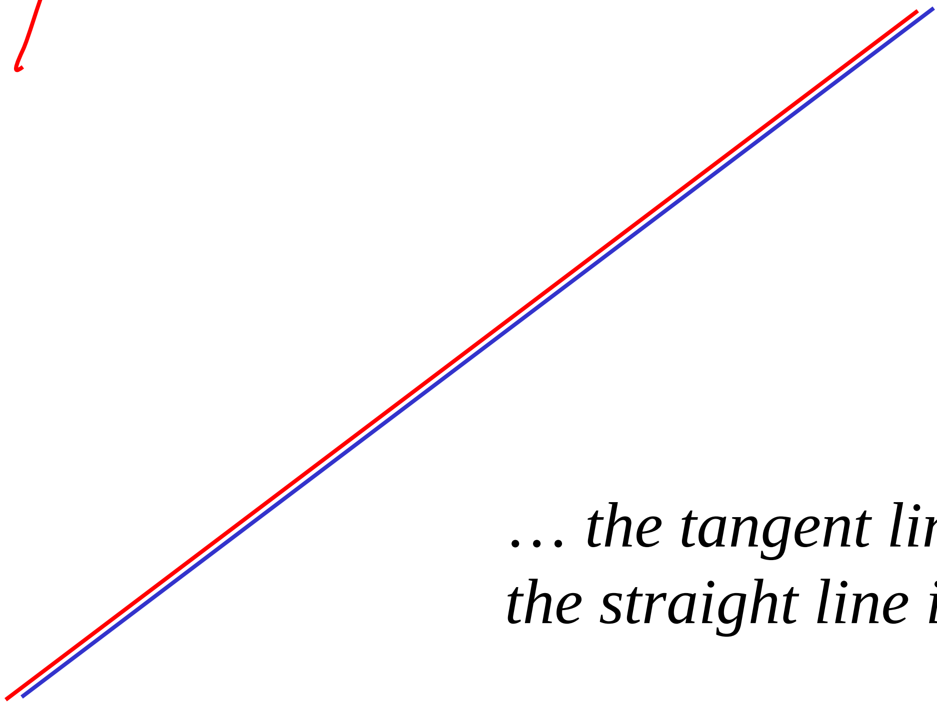
<u>A handful of counter-examples</u> - all the following straight lines should be tangent lines, too



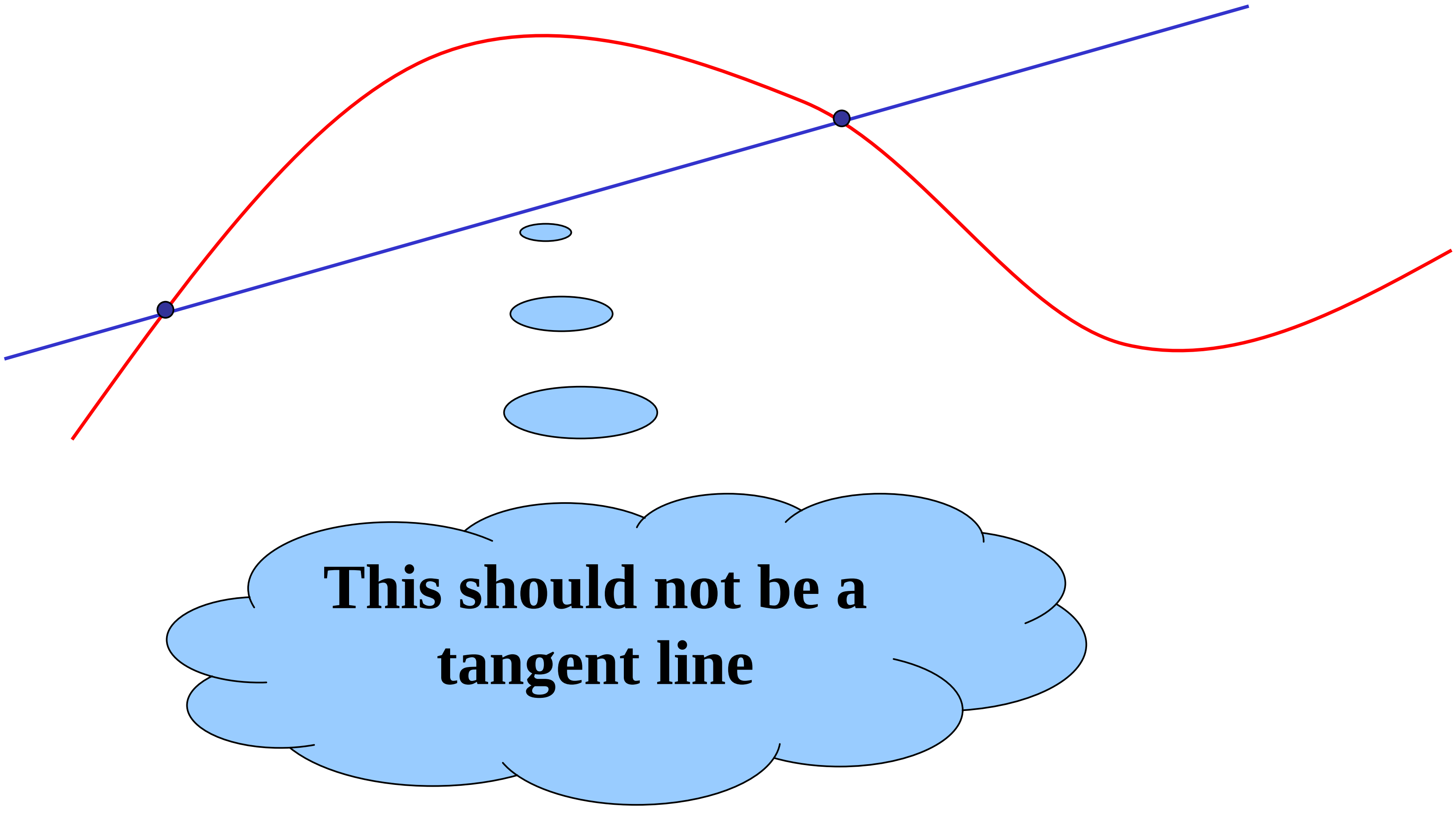
… touches the curve at more than one point …



… cuts rather than touches …



… the tangent line to a straight line is the straight line itself …


This should not be a
tangent line

$$A=\left[a,\, f\left(a\right)\right]$$

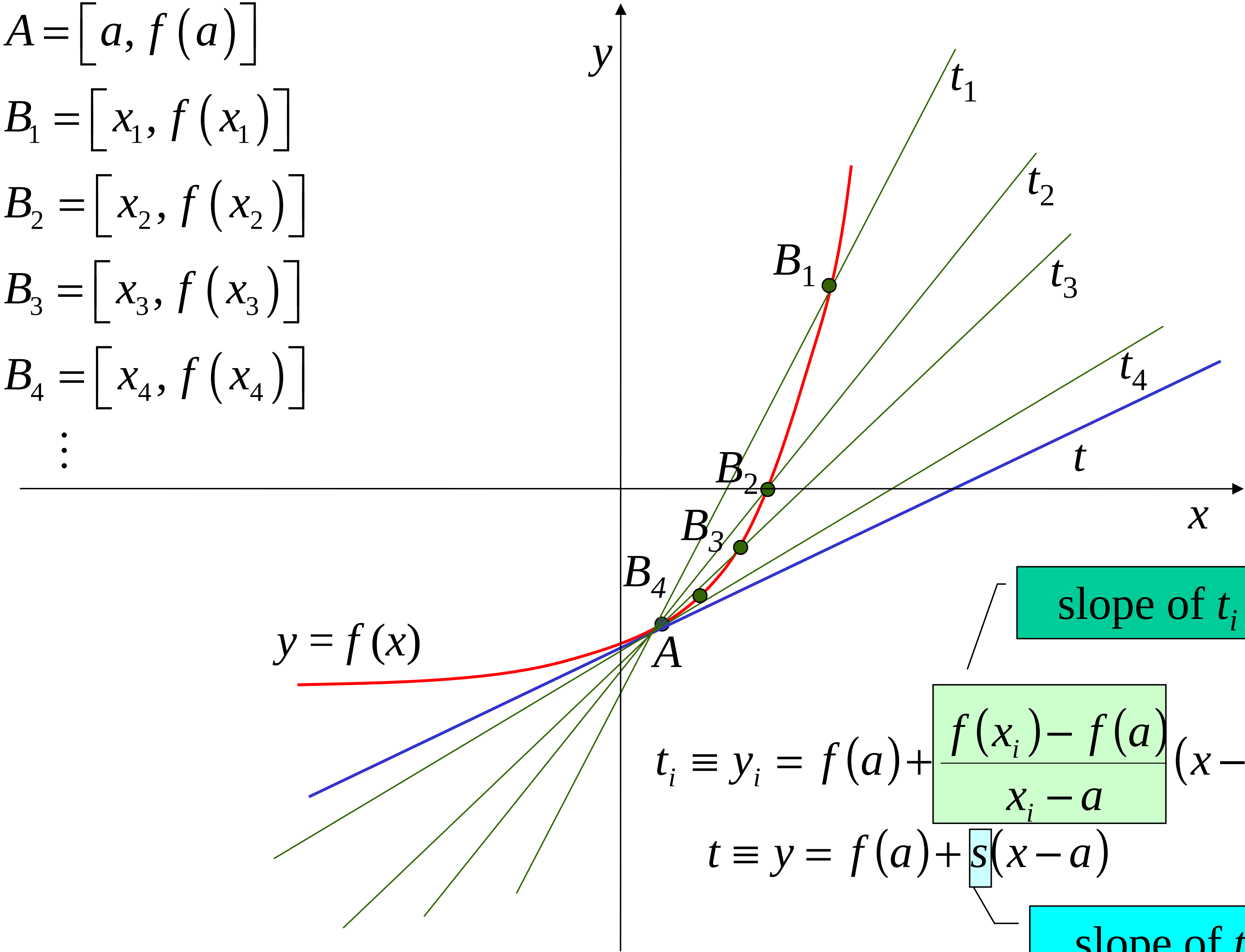
$$B_1=\left[x_1,\, f\left(x_1\right)\right]$$

$$B_2=\left[x_2,\, f\left(x_2\right)\right]$$

$$B_3=\left[x_3,\, f\left(x_3\right)\right]$$

$$B_4=\left[x_4,\, f\left(x_4\right)\right]$$

$$\vdots$$



$$t_i \equiv y_i = f\left(a\right)+\boxed{\frac{f\left(x_i\right)-f\left(a\right)}{x_i-a}}\left(x-a\right)$$

slope of t_i

$$t \equiv y = f\left(a\right)+\boxed{s}\left(x-a\right)$$

slope of t

The tangent line to a curve at a point *A* can be defined as the straight line that passes through *A* and has a slope *s* to which the following sequence of slopes approaches

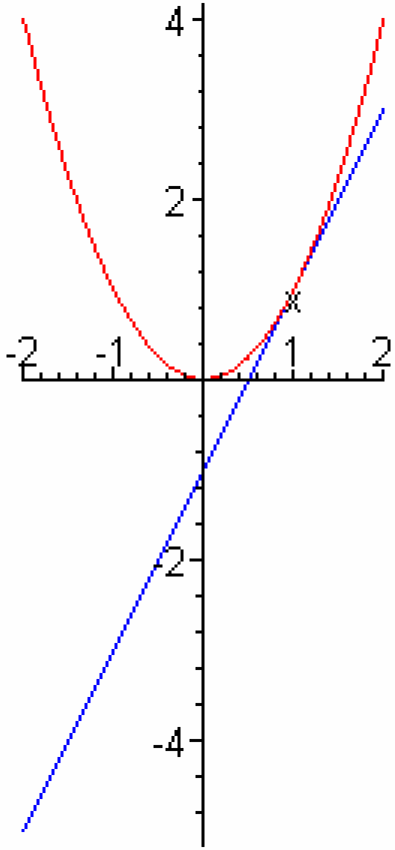
$$\frac{f(x_1)-f(a)}{x_1-a} \Rightarrow \frac{f(x_2)-f(a)}{x_2-a} \Rightarrow \frac{f(x_3)-f(a)}{x_3-a} \Rightarrow \cdots \quad s$$

as x_i approaches a.

This is formally denoted as

$$s=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

<u>Example</u> Find a tangent line t to the curve $y = x^2$ at $x = 1$.



$$y = 1 + s\left(x - 1\right) \qquad s = \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1}$$

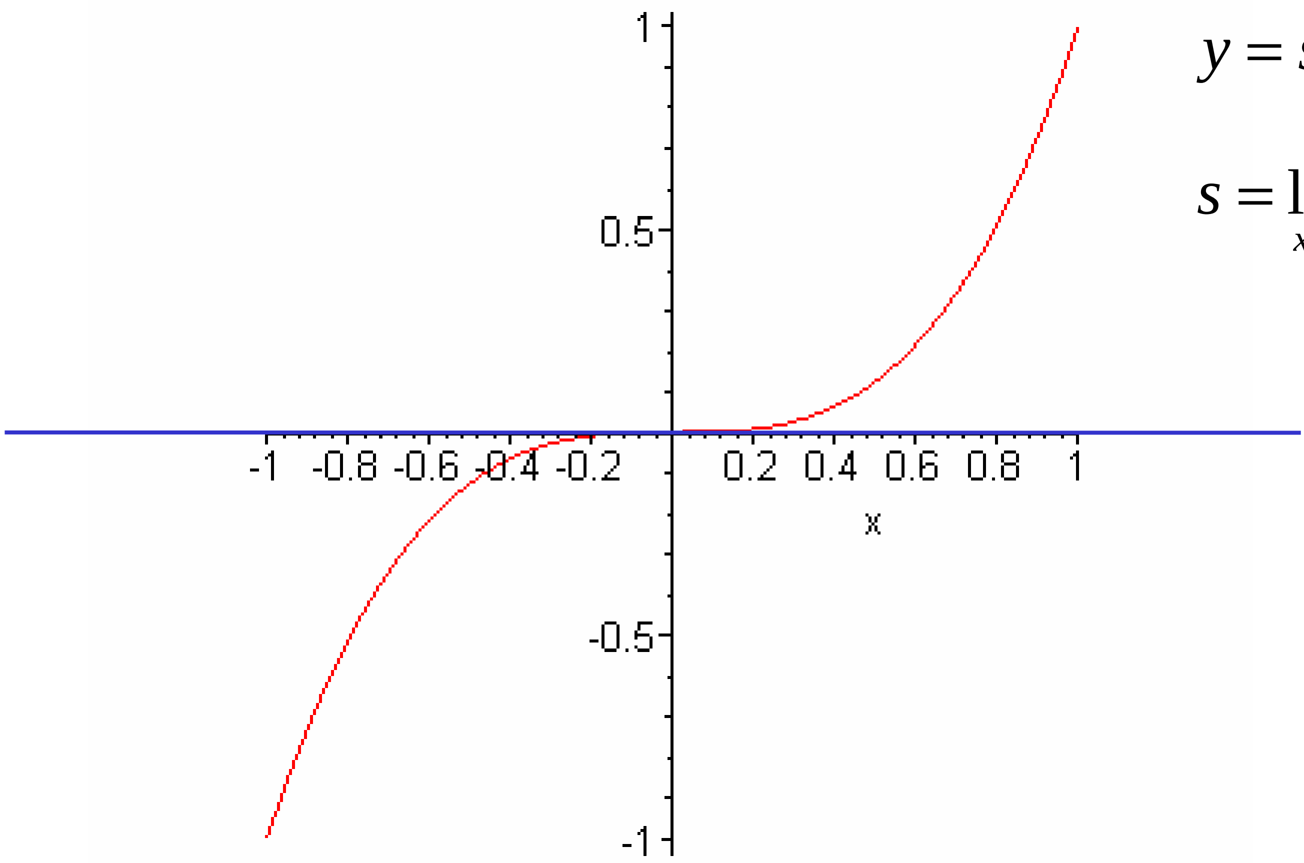
To calculate $\lim\limits_{x\to 1} \dfrac{x^2 - 1^2}{x-1}$ we cannot simply substitute 1 for x since this leads to the meaningless expression $\dfrac{0}{0}$

But everywhere, save at 1, we have

$$\frac{x^2 - 1^2}{x-1} = x+1 \quad \text{and so} \qquad \lim_{x\to 1} \frac{x^2 - 1^2}{x-1} = \lim_{x\to 1} x+1 = 2$$

$$y = 1 + 2\left(x-1\right) \quad \Rightarrow \quad y = 2x - 1$$

Example Find the tangent t to the curve $y = x^3$ at $x = 0$.



$$y = sx$$

$$s = \lim_{x \to 0} \frac{x^3 - 0}{x - 0}$$

Everywhere save at 0 we have $\dfrac{x^3}{x} = x^2$ so that $\displaystyle\lim_{x \to 0} \frac{x^3}{x} = \lim_{x \to 0} x^2 = 0$

$$y = 0$$

Example

Find the slope of the tangent line of the curve $y = x^4$ at $x = a$.

$$\lim_{x\to a}\frac{x^4 - a^4}{x-a} = \lim_{x\to a}\frac{(x-a)\left(x^3 + x^2a + xa^2 + a^3\right)}{x-a} =$$

$$= \lim_{x\to a}\left(x^3 + x^2a + xa^2 + a^3\right) = 4a^3$$

Similarly, we can prove that the slope of the tangent line to

$y = x^n$ at any point a is

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{(x-a)\left(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}\right)}{x - a} =$$

$$= \lim_{x \to a} \left(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}\right) = na^{n-1}$$

We can now view the point a at which the tangent to a function $y = f(x)$ is to be found as an independent variable and so the slope s becomes a function $s = s(a)$.

Instead of using another letter a to denote the independent variable we use also x and denote the slope function

$$y = f'(x)$$

the **<u>derivative</u>** of f

f prime of x

Other notations

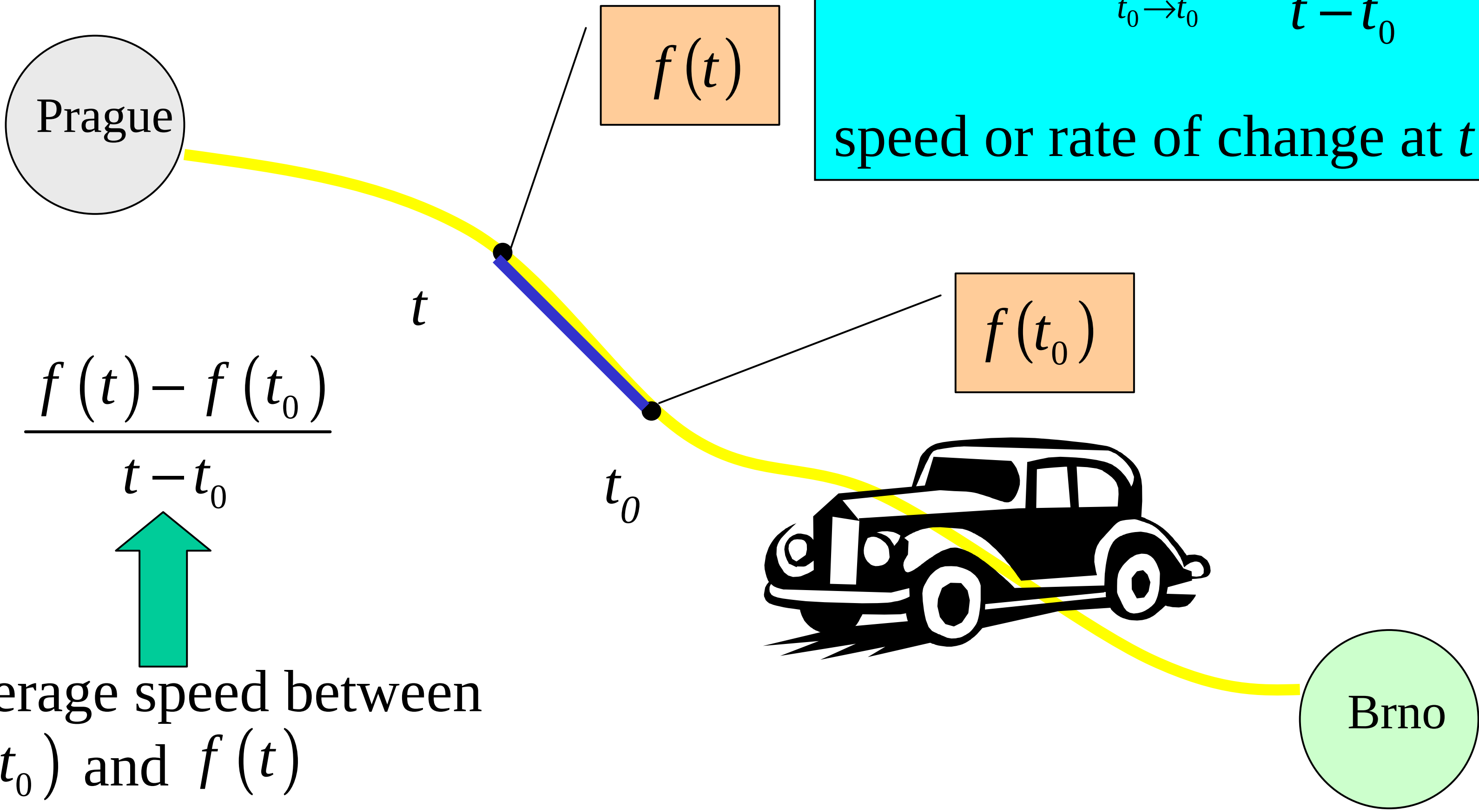
$$\frac{df}{dx} \qquad \frac{df(x)}{dx} \qquad \dot{f}(t)$$

in physics

Suppose a car starts from Brno to Prague at time 0 and and at any time t, its position on the motorway measured from Brno is given by a function $y = f(t)$.

$$f'(t) = \lim_{t_0 \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

speed or rate of change at t



$$\frac{f(t) - f(t_0)}{t - t_0}$$

average speed between
$f(t_0)$ and $f(t)$

Higher order derivatives

If $f'(x)$ has a derivative in $A \subseteq D(f)$ then this derivative is called the second derivative of $f(x)$ and denoted

$$f''(x) \text{ or } \frac{d^2 f(x)}{dx^2}$$

similarly, we can define derivatives of arbitrary orders if they exist: $f'''(x), f^{IV}(x), \ldots, f^{(n)}(x), \ldots$

$$\text{or } \frac{d^3 f(x)}{dx^3}, \frac{d^4 f(x)}{dx^4}, \ldots, \frac{d^n f(x)}{dx^n}, \ldots$$