Basic derivatives

$$\frac{dx^n}{dx} = nx^{n-1} \text{ for } n \neq -1$$

Examples

$$\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{d\sqrt[n]{x}}{dx} = \frac{1}{n\sqrt[n]{x^{n-1}}}$$

$$\frac{d1}{dx} = \frac{dx^0}{dx} = 0x^{-1} = 0$$

Basic derivatives

$$\frac{d \ln x}{dx} = \frac{1}{x}$$

Basic derivatives

$$\boxed{\frac{de^x}{dx} = e^x}$$

Basic derivatives

$$\frac{d \sin x}{dx} = \cos x$$

$$\frac{d \cos x}{dx} = -\sin x$$

Basic rules

$$\frac{d\left(f(x)\pm g(x)\right)}{dx}=\frac{df(x)}{dx}\pm\frac{dg(x)}{dx}$$

$$\frac{d\left[af(x)\right]}{dx}=a\frac{df(x)}{dx}$$

Example

$$\frac{d\left(3x^4-2x^3+x-1\right)}{dx}=3\cdot 4x^3-2\cdot 3x^2+1x^0-0=12x^3-6x^2+1$$

Basic rules

$$\boxed{\frac{df(x)g(x)}{dx} = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}}$$

Example

$$\frac{d\left(e^x x^2\right)}{dx} = \frac{de^x}{dx}x^2 + e^x\frac{dx^2}{dx} = e^x x^2 + e^x \cdot 2x = e^x\left(x^2 + 2x\right)$$

Basic rules

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{df(x)}{dx}g(x)-f(x)\frac{dg(x)}{dx}}{\left(g(x)\right)^2}$$

Example

$$\frac{d\tan x}{dx}=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x}=\frac{1}{\cos^2 x}$$

Basic rules

$$u = g(x)$$

$$h(x) = f(u)$$

$$\frac{dh(x)}{dx} = \frac{df(u)}{du}\frac{dg(x)}{dx}$$

chain rule

Examples

$$\frac{d\sin(ax)}{dx} = \cos(ax)\,a = a\cos(ax)$$

$$\frac{d}{dx}\sqrt{\ln x} = \frac{1}{2\sqrt{\ln x}}\frac{1}{x}$$

Derivative of the inverse to a function

$$y = f(x) \qquad\qquad f^{-1}(f(x)) = x$$

$$\frac{d}{dx} f^{-1}(f(x)) = \frac{df^{-1}(y)}{dy}\frac{df(x)}{dx} = 1$$

$$\boxed{\frac{df^{-1}(y)}{dy} = \frac{1}{\dfrac{df(x)}{dx}}}$$

$$\boxed{\frac{df^{-1}(x)}{dx} = \frac{1}{\dfrac{df(y)}{dy}}}$$

Examples

$$y = \arcsin x \qquad \frac{d \arcsin x}{dx} = \frac{1}{\cos y} = \frac{1}{\cos\left(\arcsin x\right)} =$$

$$= \frac{1}{\sqrt{1 - \sin^2\left(\arcsin x\right)}} = \frac{1}{\sqrt{1 - x^2}}$$

$$y = \ln x \qquad \frac{d \ln x}{dx} = \frac{1}{\frac{de^y}{dy}} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

- $$\frac{d\arctan x}{dx}=\frac{1}{1+x^2}$$

- $$\frac{d\arccos x}{dx}=\frac{-1}{\sqrt{1-x^2}}$$

- $$\frac{d\,\mathrm{arccot}\,x}{dx}=\frac{-1}{1+x^2}$$

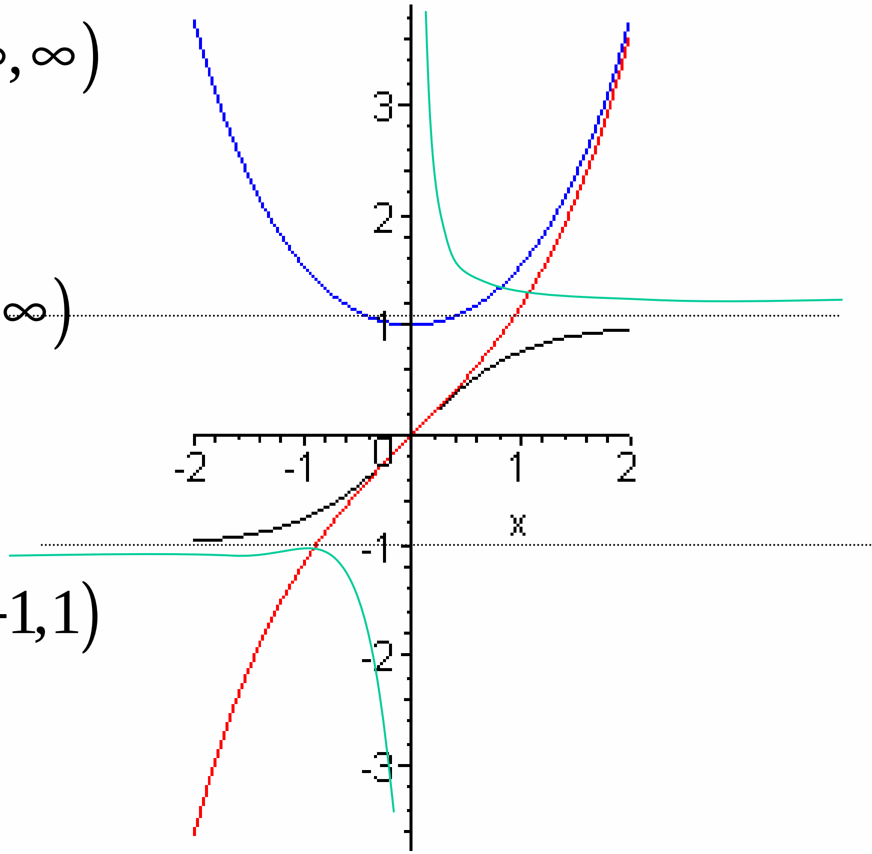
Hyperbolic trigonometric functions

hyperbolic sine

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \quad D = (-\infty, \infty), R = (-\infty, \infty)$$

hyperbolic cosine

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \quad D = (-\infty, \infty), R = (1, \infty)$$



hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} \quad D = (-\infty, \infty), R = (-1, 1)$$

hyperbolic cotangent

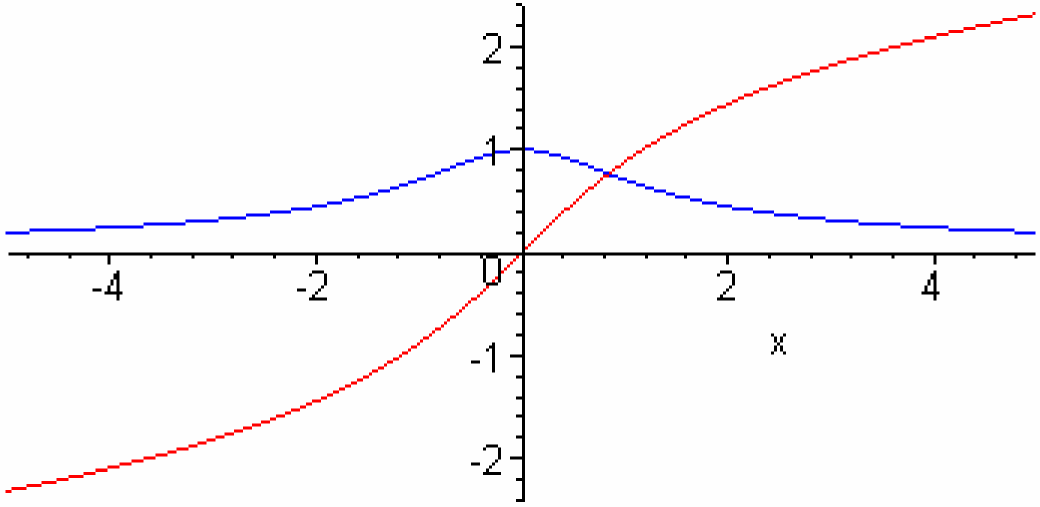
$$\coth x = \frac{\cosh x}{\sinh x} \quad D = (-\infty, 0) \cup (0, \infty), R = (-\infty, -1) \cup (1, \infty)$$

- $$\frac{d\sinh x}{dx} = \cosh x$$
- $$\frac{d\cosh x}{dx} = \sinh x$$
- $$\frac{d\tanh x}{dx} = \frac{1}{\cosh^2 x}$$
- $$\frac{d\coth x}{dx} = \frac{-1}{\sinh^2 x}$$

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

<u>Inverse hyperbolic sine</u>

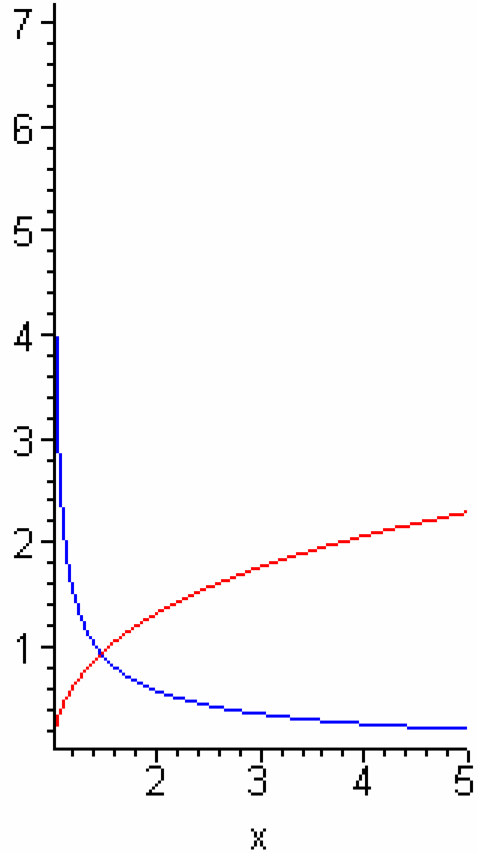
$y = \operatorname{arcsinh} x \quad D(f) = R(f) = (-\infty, \infty) \qquad y = \ln\left(x + \sqrt{x^2+1}\right)$



$$\frac{d \operatorname{arcsinh} x}{dx} = \frac{1}{\sqrt{x^2+1}}$$

Inverse hyperbolic cosine

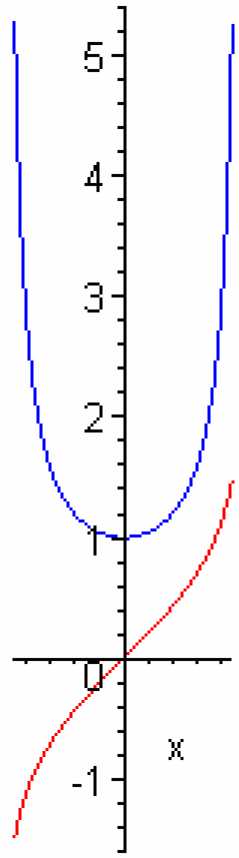
$$y = \operatorname{arccosh} x \quad D(f) = (1, \infty), R(f) = (0, \infty) \quad y = \ln\left(x + \sqrt{x^2 - 1}\right)$$



$$\frac{d \operatorname{arccosh} x}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

Inverse hyperbolic tangent

$$y = \operatorname{arctanh} x \quad D(f) = (-1,1), R(f) = (-\infty, \infty) \quad y = \ln\sqrt{\left|\frac{1+x}{1-x}\right|}$$

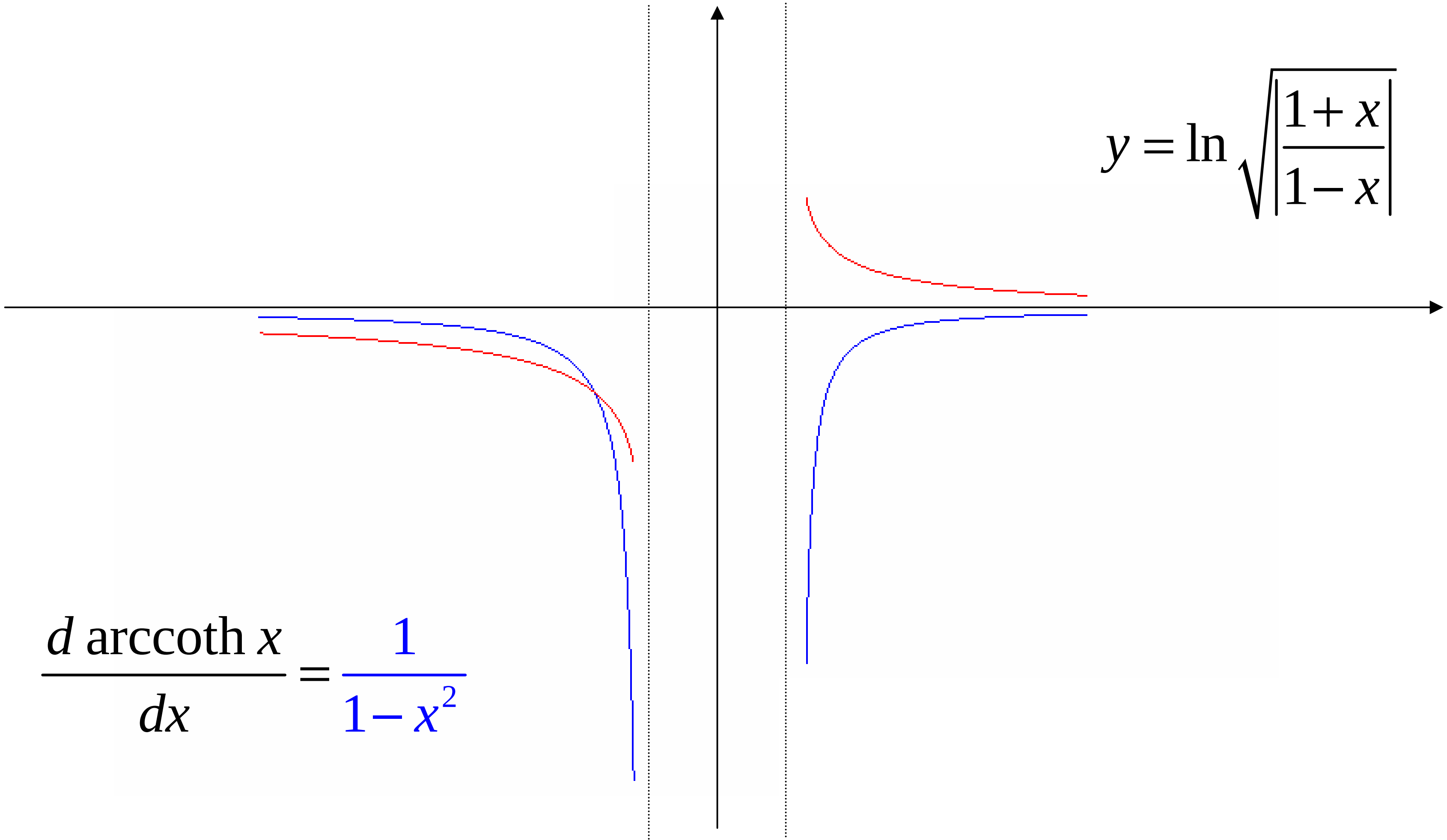


$$\frac{d \operatorname{arctanh} x}{dx} = \frac{1}{1-x^2}$$

Inverse hyperbolic cotangent

$$y = \operatorname{arccoth} x \quad D(f) = (-\infty, 1) \cup (1, \infty),\ R(f) = (-\infty, \infty) - \{0\}$$

$$y = \ln \sqrt{\left|\frac{1+x}{1-x}\right|}$$



$$\frac{d \operatorname{arccoth} x}{dx} = \frac{1}{1-x^2}$$

Problems to solve

$$y = \ln\cos\frac{x-1}{x} \qquad y' = \frac{-1}{x^2}\tan\frac{x-1}{x}$$

$$y = \sqrt{x^2+1} - \ln\frac{1+\sqrt{x^2+1}}{x} \qquad y' = \frac{\sqrt{x^2+1}}{x}$$

$$y = \ln\frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}} + 2\arctan\sqrt{\sin x} \qquad y' = \frac{2}{\cos x\sqrt{\sin x}}$$

Logarithmic derivative

The logarithmic derivative of a function $y = f(x)$ is defined as the derivative of the logarithm of that function, that is

$$\frac{d \ln y}{dx} = \frac{1}{y}\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Taking first the logarithm of a function and then differentiating may sometimes help find the derivative of the function.

Example

Find the derivative of the function $y = x^x$

Taking logarithm

$\ln y = x \ln x$

differentiating

$$\frac{y'}{y} = \ln x + \frac{x}{x} = \ln x + 1$$

yields

$$y' = y(\ln x + 1) = x^x(\ln x + 1)$$

Example

Calculate the derivative of $y=\sqrt[3]{x^2}\,\frac{1-x}{1+x^2}\sin^3 x\cos^2 x$

$$y=\sqrt[3]{x^2}\,\frac{1-x}{1+x^2}\sin^3 x\cos^2 x$$

$$\ln y=\frac{2}{3}\ln x+\ln(1-x)-\ln\left(1+x^2\right)+3\ln\sin x+2\ln\cos x$$

$$\frac{y'}{y}=\frac{2}{3x}-\frac{1}{1-x}-\frac{2x}{1+x^2}+3\frac{\cos x}{\sin x}-2\frac{\sin x}{\cos x}$$

$$y'=\left(\sqrt[3]{x^2}\,\frac{1-x}{1+x^2}\sin^3 x\cos^2 x\right)\left(\frac{2}{3x}-\frac{1}{1-x}-\frac{2x}{1+x^2}+3\cot x-2\tan x\right)$$