Some functions and their derivatives may be difficult to evaluate, however, at special points, this may be easy.

$$\sin x \quad \to \quad \cos x \to \quad -\sin x \to \quad -\cos x \to \quad \sin x \to \quad \ldots$$

$$\text{at } \frac{\pi}{4}: \quad \frac{\sqrt{2}}{2} \quad \to \quad \frac{\sqrt{2}}{2} \quad \to \frac{-\sqrt{2}}{2} \quad \to \quad \frac{-\sqrt{2}}{2} \quad \to \quad \ldots$$

$$e^x \quad \to \quad e^x \to \quad e^x \to \quad e^x \to \quad e^x \to \quad \ldots$$

$$\text{at } 0: \quad 1 \quad \to \quad 1 \quad \to 1 \quad \to \quad 1 \quad \to \quad \ldots$$

For a function $f(x)$ that has derivatives of order *n* at a point *a*,we want to find a polynomial $T_{f,n,a}(x)$ that approximates it in an interval $[a,x]$ or$[x,a]$.

For such an approximation to be good, we need the first *n* derivaties of $T_{f,n,a}(x)$ and $f(x)$ to be identical.

This means that

$$T_{f,n,a}(x) = f(a) + \frac{f^{'}(a)}{1!}(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$T_{f,n,a}(x)$ is called the *n*-th order Taylor polynomial of $f(x)$ at *a*

Given a function $f(x)$, points a and x, and an intereger n, how good the approximation $T_{f,n,a}(x)$ is at x?

Taylor theorem

Let a, x be two different numbers, $a < x$ or $x < a$ and $n \geq 0$ an integer. Let $f(x)$ be a function that has derivatives of up to order $n+1$ in $[a,x]$. Let us define

$$R_{n+1}(x) = f(x) - T_{f,n,a}(x)$$

Then there exists a number ξ, $a < \xi < x$ if $x < a$ and $x < \xi < a$ if $x < a$

such that

$$R_{n+1}(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi).$$

Note that we do not know the exact value of $R_{n+1}(x)$ since we do not know the exact position of ξ between a and x. However, in many cases, we can find an estimate of its absolute value.

<u>Example</u>

Find the value $\sin\left(\frac{\pi}{2}-0.1\right)$ with a precision better than 10^{-5}

We have

$$\sin\left(\frac{\pi}{2}-0.1\right)=\sin\left(\frac{\pi}{2}\right)+\frac{\cos\left(\frac{\pi}{2}\right)}{1!}(-0.1)+\frac{-\sin\left(\frac{\pi}{2}\right)}{2!}(-0.1)^2+$$

$$+\frac{-\cos\left(\frac{\pi}{2}\right)}{3!}(-0.1)^3+\frac{\sin\left(\frac{\pi}{2}\right)}{4!}(-0.1)^4+\dots$$

$$\sin\left(\frac{\pi}{2}-0.1\right)=1-\frac{0.1^2}{2!}+\frac{0.1^4}{4!}-\cdots+(-1)^k\frac{(0.1)^{2k}}{(2k)!}+$$

$$+(-1)^{k+1}\frac{\sin^{(2k+1)}(\xi)}{(2k+1)!}-(0.1)^{2k+1}$$

We need $\left|(-1)^{k+1}\frac{\sin^{(2k+1)}(\xi)}{(2k+1)!}(-0.1)^{2k+1}\right|$ to be less than 10^{-5}

However, since $\left|\sin^{(2k+1)}(\xi)\right|\le 1$, we can write

$$\left|(-1)^{k+1}\frac{\sin^{(2k+1)}(\xi)}{(2k+1)!}(-0.1)^{2k+1}\right| \le \left|\frac{1}{(2k+1)!}0.1^{2k+1}\right| < \frac{1}{10^5}$$

$$\left|\frac{1}{(2k+1)!}0.1^{2k+1}\right| < \frac{1}{10^5} \qquad \Rightarrow \qquad \left|\frac{1}{(2k+1)!10^{2k+1}}\right| < \frac{1}{10^5}$$

$$\Rightarrow \quad \left|(2k+1)!10^{2k+1}\right| > 10^5$$

For $k = 1$ we have $6!10^3 < 10^5$ while $k = 2$ gives $120 \cdot 10^5 > 10^5$

so that

$$\sin\left(\frac{\pi}{2} - 0.1\right) \approx 1 - \frac{0.1^2}{2!} + \frac{0.1^4}{4!} 0.99500$$

MacLaurin polynomial

The MacLaurin polynomial of a function $f(x)$ is its Taylor polynomial at $a = 0$ so that:

$$T_{f,n,0}(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

MacLaurin polynomials to remember

- $f(x)=e^x \quad T_{f,n,0}=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!} \qquad x\in R$
- $f(x)=\sin x \quad T_{f,n,0}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^{k+1}\frac{x^{2k+1}}{n!} \qquad x\in R$
- $f(x)=\cos x \quad T_{f,n,0}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^{k}\frac{x^{2k}}{n!} \qquad x\in R$
- $f(x)=\ln(1+x) \quad T_{f,n,0}=\frac{x}{1}-\frac{x^2}{2}+\cdots+(-1)^{n+1}\frac{x^n}{n} \qquad -1<x<1$

If a function $f(x)$ has a Taylor polynomial, $T_{f,n,a}(x)$, then it is unique.

This means that if, for a given function, we can find a polynomial that has the form of a Taylor polynomial to, we know that it is the Taylor polynomial to that function.

Example

$$f\left(x\right)=\frac{1}{1-x}$$

For $-1 < x < 1$, $f(x)$ is the sum of an infinite geometric sequence $1, x, x^2, x^3, \ldots$ so that

$$\frac{1}{1-x}=1+x+x^2+\cdots$$

This means that

$$T_{f,n,0}=1+x+x^2+\cdots+x^n \text{ for } -1<x<1$$

Example

$$f(x) = \cos^2 x \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\cos 2x \approx 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \cdots + (-1)^k \frac{2^{2k} x^{2k}}{(2k)!}$$

$$\cos^2 x \approx 1 - \frac{2x^2}{2!} + \frac{8x^4}{4!} - \cdots + (-1)^k \frac{2^{2k-1} x^{2k}}{(2k)!}$$

Example

$$f(x) = \arctan x \qquad f'(x) = \frac{1}{1+x^2}$$

$$f'(x) \approx 1 - x^2 + x^4 - \cdots + (-1)^{k+1} x^{2k}$$

Since for the function $T(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{k+1} x^{2k+1}$

we have $T'(x) = 1 - x^2 + x^4 - \cdots + (-1)^{k+1} x^{2k}$

$T(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{k+1} x^{2k+1}$ must be the Taylor polynomial

for $f(x) = \arctan x$

One way of calculating π

$$\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{2k+1}\frac{x^{2k+1}}{2k+1}$$

$$\arctan 1 = \frac{\pi}{4}$$

$$\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

For practical purposes, however, this method is of little use.

Example

$$f\left(x\right)=\ln\frac{1+x}{1-x}$$

$$f\left(x\right)=\ln\left(1+x\right)-\ln\left(1-x\right)$$

$$f(x)\approx x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n+1}\frac{x^n}{n}+x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}$$

$$f(x)\approx 2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots+\frac{x^{2k-1}}{2k-1}\right)$$

The formula $f(x) \approx 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2k-1}}{2k-1} \right)$

can be used, for example, to evaluate natural logarithms

$$\frac{1+x}{1-x} = a \quad \Rightarrow \quad x = \frac{a-1}{a+1} \quad \Rightarrow \quad -1 < x < 1 \text{ for } a > 0$$

Example

Calculate ln 2 to five decimals.

For $x=\dfrac{1}{3}$ we have $\dfrac{1+1/3}{1-1/3}=2$

It can be further established that $\left|R_{n+1}(x)\right|\le\dfrac{2}{(n+1)3^{n+1}}$

$$\frac{2}{(n+1)3^{n+1}}<\frac{1}{10^5}\quad\Rightarrow\quad(n+1)3^{n+1}>2\cdot10^5$$

This means that we have to take *n* at least 9 and so

$$\ln 2\approx 2\left(\frac{1}{3}+\frac{(1/3)^3}{3}+\frac{(1/3)^5}{5}+\frac{(1/3)^7}{7}+\frac{(1/3)^9}{9}\right)=0.69314$$

For $|x|<1$ and any real a we can write

$$(x+y)^a = x^a + \binom{a}{1}x^{a-1}y + \binom{a}{2}x^{a-2}y^2 + \binom{a}{3}x^{a-3}y^3 + \cdots$$

where $\binom{a}{k} = \dfrac{a(a-1)(a-2)\cdots(a-k+1)}{k!}$

This can be used, for example, when calculating the Taylor polynomial for $f(x) = \sqrt{1+x}$

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \binom{1/2}{1} x + \binom{1/2}{2} x^2 + \cdots + \binom{1/2}{n} x^n + \cdots$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{1}{2^2 2!} x^2 + \frac{1 \cdot 3}{2^3 3!} x^3 + \cdots + (-1)^{n+1} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-3)}{2^n n!} + \cdots$$