If a function $y = f(x)$ has a derivative at a,

then it is continuous at a.

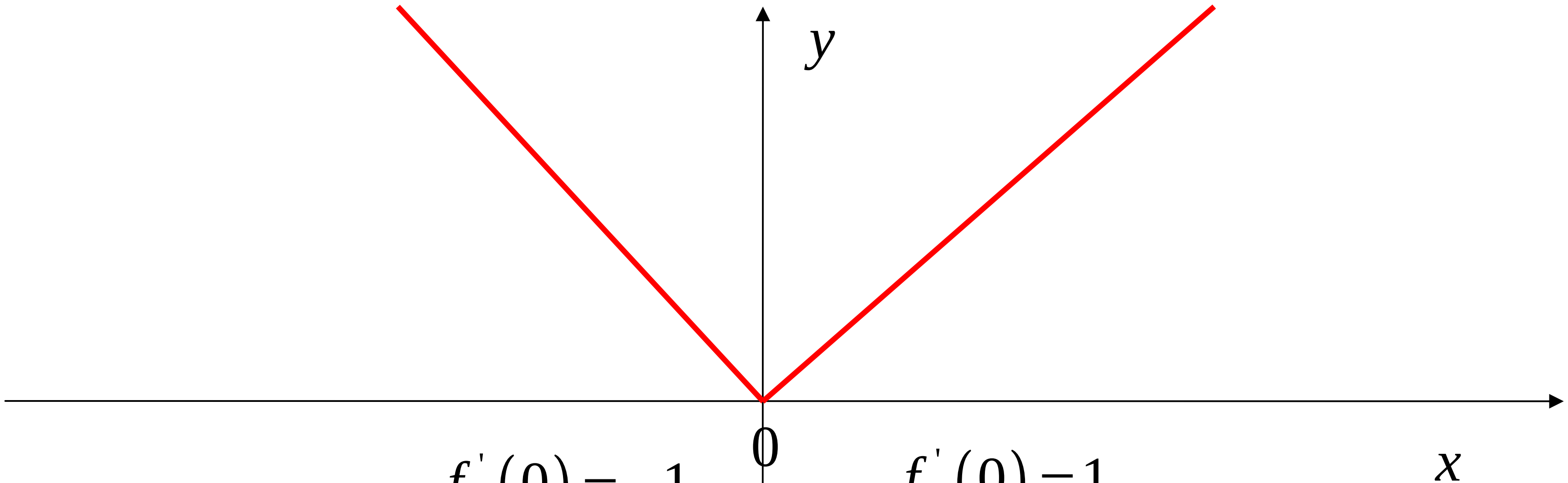
Proof

If $f^{'}(a)$ exists, then the limit $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists and so the function value $f(a)$ of $f(x)$ at a exists.

$$\lim_{x \to a} f(x) = \lim_{x \to a} f(x) - f(a) + f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) + f(a) =$$

$$= \lim_{x \to a} f^{'}(a)(x - a) + f(a) = f(a)$$

The reverse assertion of the preceding theorem is not true, that is, if a function is continuous at *a*, it need not have a derivative at *a*.



$f'_{-}(0) = -1$ $\qquad$ $f'_{+}(0) = 1$

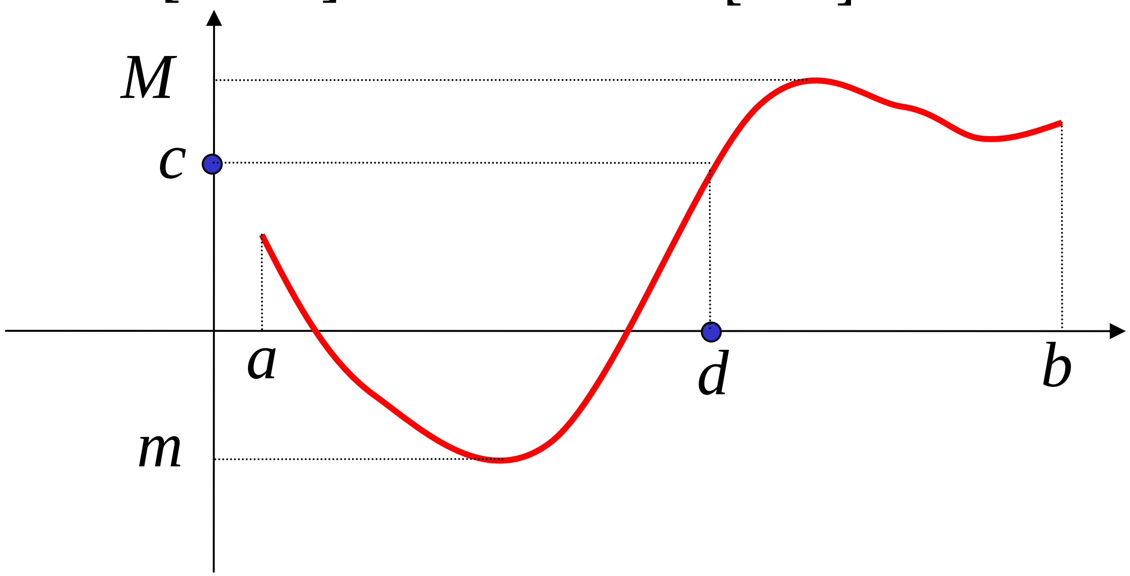
$f'(0)$ **does not exist**

- If a function $f(x)$ is continuous at every point $x \in (a,b)$ we say that it is continuous over the open interval (a,b). If, moreover, it is continuous on the right at a and on the left at b, we say that it is continuous over the closed interval $[a,b]$.

- If a constant K exists such that $|f(x)| \le K$ for every $x \in [a,b]$, we say that $f(x)$ is bounded in $[a,b]$.

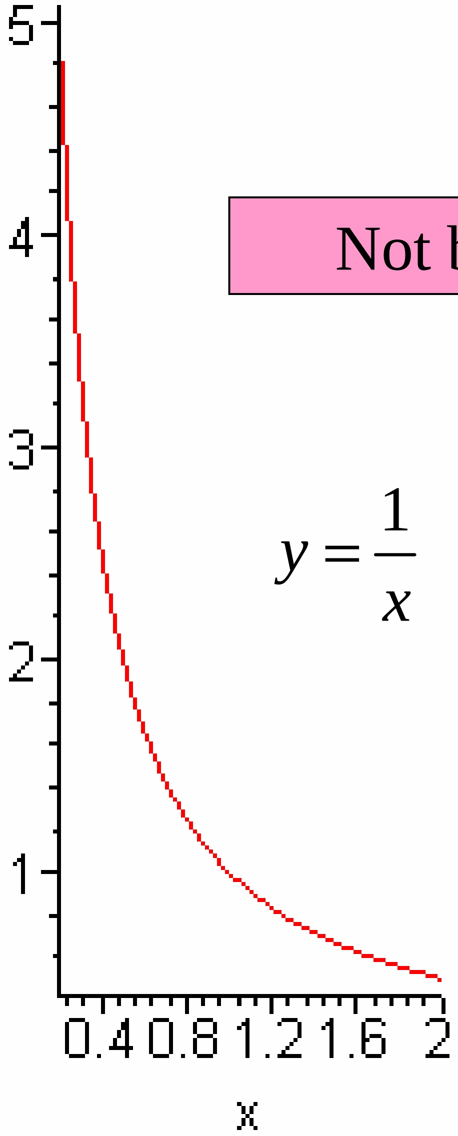
If a function $f(x)$ is continuous over a closed interval $[a,b]$, then

- it is bounded in $[a,b]$
- it reaches in $[a,b]$ its maximum and minimum values M,m
- for any $c \in [m,M]$ there is a $d \in [a,b]$ such that $f(d)=c$



Counterexample

Not defined and thus not continuous at 0



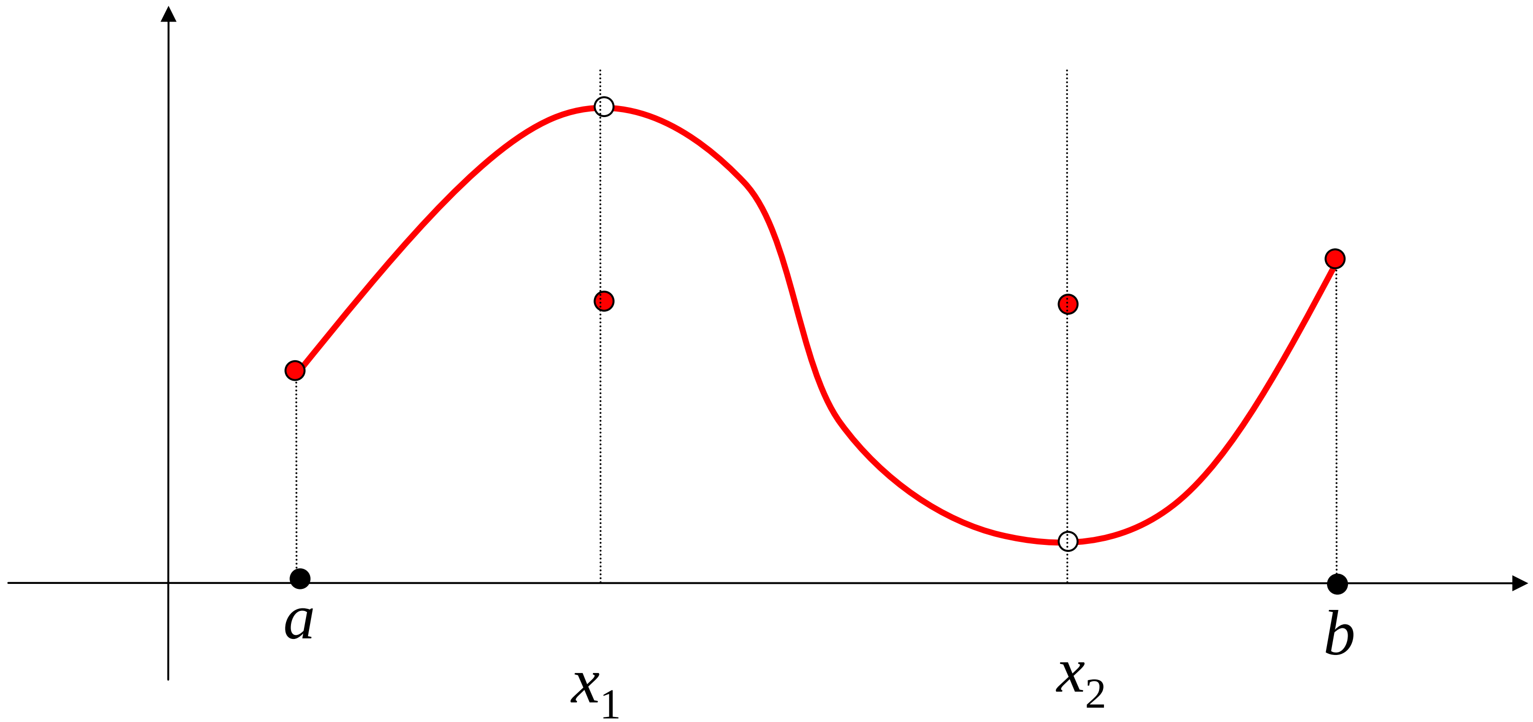
Not bounded

$$y = \frac{1}{x}$$

Counterexample

Not continuous at x_1 and x_2

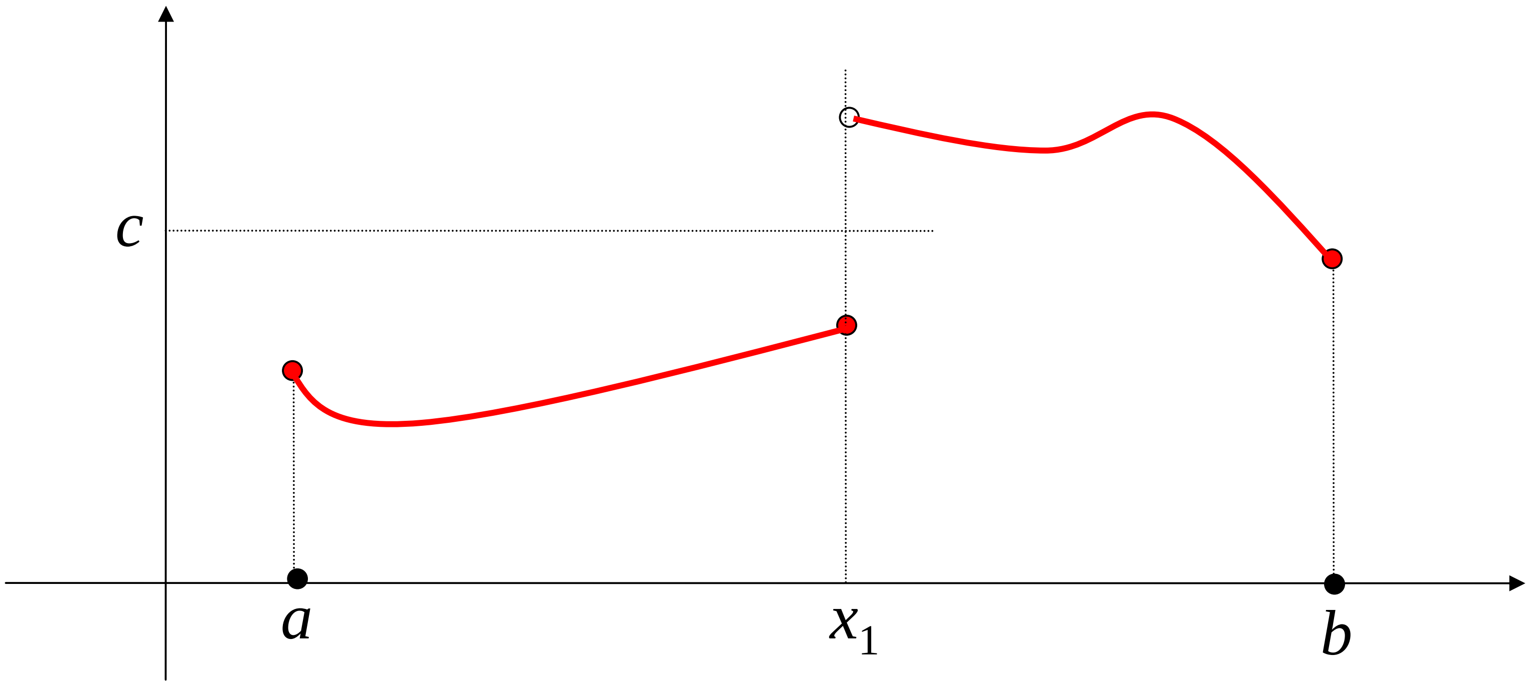
No maximum or minimum value



<u>Counterexample</u>

Not continuous at x_1

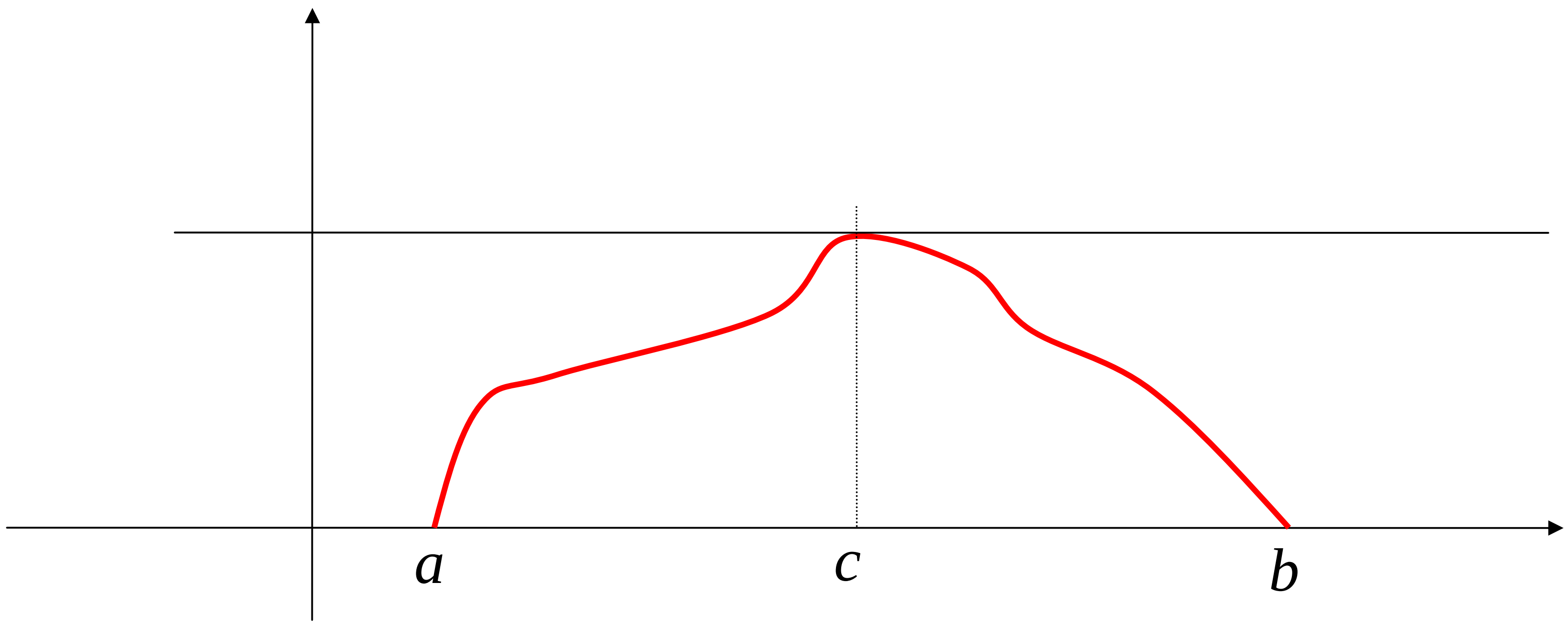
Value c not reached



Rolle's theorem

Let a, b be two numbers, $a < b$. Let $f(x)$ be a function which is continuous over the closed interval $[a,b]$ and has a derivative for every $x \in (a,b)$. Assume that $f(a) = f(b) = 0$.

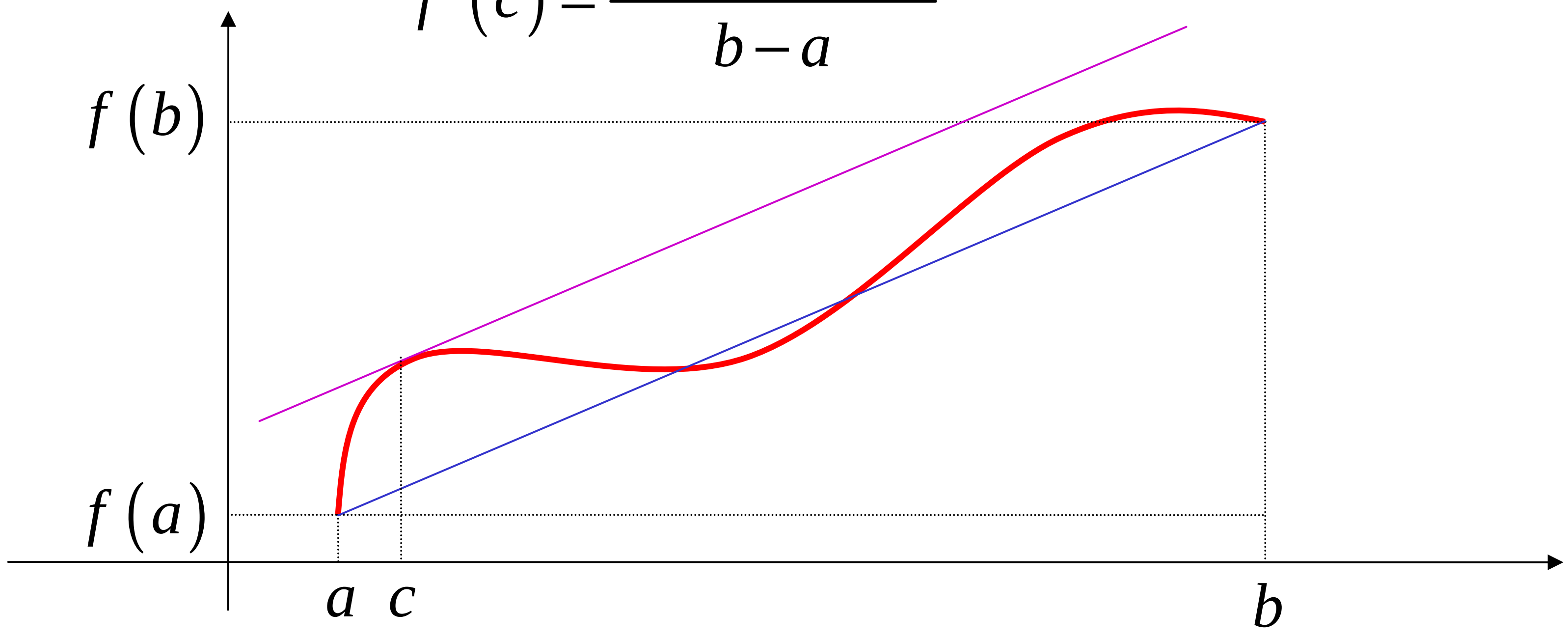
Then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.



The mean value theorem

Let a, b be two numbers, $a < b$. Let $f(x)$ be a function which is continuous over the closed interval $[a,b]$ and has a derivative for every $x \in (a,b)$. Then there exists a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



If $f'(x) > 0$ for $x \in (a,b)$ and $f(x)$ is continuous over $[a,b]$, then $f(x)$ is strictly increasing in $[a,b]$.

Let $x_1, x_2 \in [a,b]$ and suppose $x_1 < x_2$. By the mean value theorem, there exists a point c such that $x_1 < c < x_2$ and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$$

$c \in (a,b)$ means that $f'(c) > 0$ and since $x_2 - x_1 > 0$, we have

$$f(x_2) - f(x_1) > 0 \Rightarrow f(x_2) > f(x_1)$$

If $f'(x) < 0$ for $x \in (a,b)$ and $f(x)$ is continuous over $[a,b]$, then $f(x)$ is strictly decreasing in $[a,b]$.

Let $x_1, x_2 \in [a,b]$ and suppose $x_1 < x_2$. By the mean value theorem, there exists a point c such that $x_1 < c < x_2$ and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$$

$c \in (a,b)$ means that $f'(c) < 0$ and since $x_2 - x_1 > 0$, we have

$$f(x_2) - f(x_1) < 0 \Rightarrow f(x_2) < f(x_1)$$