In the below theorem, "lim" will mean any of the following limits

$\lim\limits_{x\to a}$ $\lim\limits_{x\to a+}$ $\lim\limits_{x\to a-}$ $\lim\limits_{x\to\infty}$ $\lim\limits_{x\to-\infty}$

Let either $\lim f(x)=\lim g(x)=0$ or $\lim\left|g(x)\right|=\infty$. Then the following is true:

if the limit $\lim\dfrac{f'(x)}{g'(x)}$ exists (proper or improper one), then also the limit $\lim\dfrac{f(x)}{g(x)}$ exists and we have

$$\lim\frac{f(x)}{g(x)}=\lim\frac{f'(x)}{g'(x)}$$

The preceding theorem may be proved using the mean value theorem and is a basis for what is called L'Hospital's rule, which is a useful tool for evaluating limits of the type

$$\frac{0}{0} \quad \frac{\infty}{\infty} \quad 0.\infty \quad \infty - \infty \quad 1^{\infty} \quad 0^{0} \quad \infty^{0} \text{etc.}$$

Here $\frac{\mathbf{0}}{\mathbf{0}}$ for example means that we are to calculate the limit

$$\lim \frac{f(x)}{g(x)} \quad \text{where} \quad \lim f(x) = \lim g(x) = 0$$

$\frac{0}{0}$ example

$$\lim_{x\to 0}\frac{x\cos x-\sin x}{x^3}=\lim_{x\to 0}\frac{\cos x-x\sin x-\cos x}{3x^2}=\lim_{x\to 0}\frac{-\sin x}{3x}=\frac{-1}{3}$$

$\frac{0}{0}$ example

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x-\sin x}=\lim_{x\to 0}\frac{\frac{1}{\cos^2 x}-\cos x}{1-\cos x}=\lim_{x\to 0}\frac{1-\cos^3 x}{\cos^2 x(1-\cos x)}=$$

$$=\lim_{x\to 0}\frac{3\cos^2 x\sin x}{-2\cos x\sin x(1-\cos x)+\cos^2 x\sin x}=$$

$$=\lim_{x\to 0}\frac{3\cos^2 x\sin x}{-2\cos x\sin x+3\cos^2 x\sin x}=\lim_{x\to 0}\frac{3\cos^2 x}{-2\cos x+3\cos^2 x}=3$$

Note that we also have

$$\lim_{x\to 0}\frac{1-\cos^3 x}{\cos^2 x(1-\cos x)}=\lim_{x\to 0}\frac{1+\cos x+\cos^2 x}{\cos^2 x}=3$$

$\frac{\infty}{\infty}$ example

$$\lim_{x\to\infty}\frac{e^x}{x^5}=\lim_{x\to\infty}\frac{e^x}{5x^4}=\lim_{x\to\infty}\frac{e^x}{5\cdot 4x^3}=\lim_{x\to\infty}\frac{e^x}{5\cdot 4\cdot 3x^2}+\lim_{x\to\infty}\frac{e^x}{5\cdot 4\cdot 3\cdot 2x}=$$

$$=\lim_{x\to\infty}\frac{e^x}{5\cdot 4\cdot 3\cdot 2\cdot 1}=\infty$$

$0 \cdot \infty$ example

$$\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}=\lim_{x\to 1}\frac{(1-x)}{\cot\frac{\pi x}{2}}=\lim_{x\to 1}\frac{-1}{\frac{-\pi}{2\sin^2\frac{\pi x}{2}}}=\lim_{x\to 1}\frac{2\sin^2\frac{\pi x}{2}}{\pi}=\frac{2}{\pi}$$

$\infty - \infty$ example

$$\lim_{x\to\infty} \sqrt{2x+1} - \sqrt{x} = \lim_{t\to 0+} \sqrt{\frac{2}{t}+1} - \sqrt{\frac{1}{t}} = \lim_{t\to 0+} \frac{\sqrt{2+t}-1}{\sqrt{t}} = \infty$$

1^{∞} example

$$\lim_{x\to 0}\left(1+x^2\right)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln\left(1+x^2\right)^{\frac{1}{x}}} = \lim_{x\to 0} e^{\frac{\ln\left(1+x^2\right)}{x}} =$$

$$= \lim_{x\to 0} e^{\frac{\frac{2x}{1+x^2}}{1}} = \lim_{x\to 0} e^{\frac{2x}{1+x^2}} = e^0 = 1$$

∞^0 example

$$\lim_{x\to\infty}\sqrt[x]{x}=\lim_{x\to\infty}x^{\frac{1}{x}}=\lim_{x\to\infty}e^{\frac{\ln x}{x}}=\lim_{x\to\infty}e^{\frac{\frac{1}{x}}{1}}=\lim_{x\to\infty}e^{\frac{1}{x}}=e^0=1$$

0^0 example

$$\lim_{x\to 0} x^{\sin x} = \lim_{x\to 0} e^{\ln x \sin x} = \lim_{x\to 0} e^{\frac{\sin x}{\frac{1}{\ln x}}} = \lim_{x\to 0} e^{\frac{\cos x}{\frac{-1}{x\ln^2 x}}} = e^{\lim\limits_{x\to 0} -x\cos x \ln^2 x} =$$

$$= e^{-\lim\limits_{x\to 0}\cos x \lim\limits_{x\to 0} x\ln^2 x} = e^{-\lim\limits_{x\to 0} x\ln^2 x} = e^{-\lim\limits_{x\to 0}\frac{\ln^2 x}{\frac{1}{x}}} =$$

$$= e^{-\lim\limits_{x\to 0}\frac{\frac{2\ln x}{x}}{\frac{-1}{x^2}}} = e^{\lim\limits_{x\to 0} 2x\ln x} = e^{\lim\limits_{x\to 0}\frac{2\ln x}{\frac{1}{x}}} =$$

$$= e^{\lim\limits_{x\to 0}\frac{\frac{2}{x}}{\frac{-1}{x^2}}} = e^{-\lim\limits_{x\to 0} 2x} = e^0 = 1$$

$$y = x^{\sin x}$$