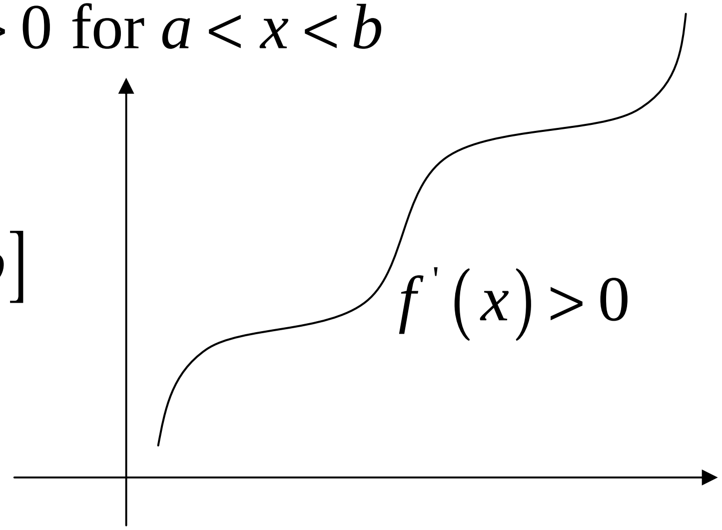
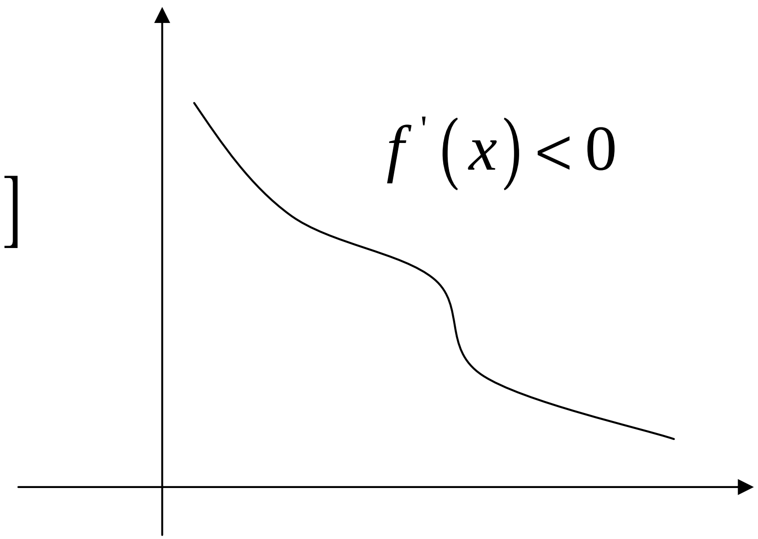
If $f(x)$ is defined on $[a,b]$ and $f^{'}(x) > 0$ for $a < x < b$

then $f(x)$ is increasing on $[a,b]$



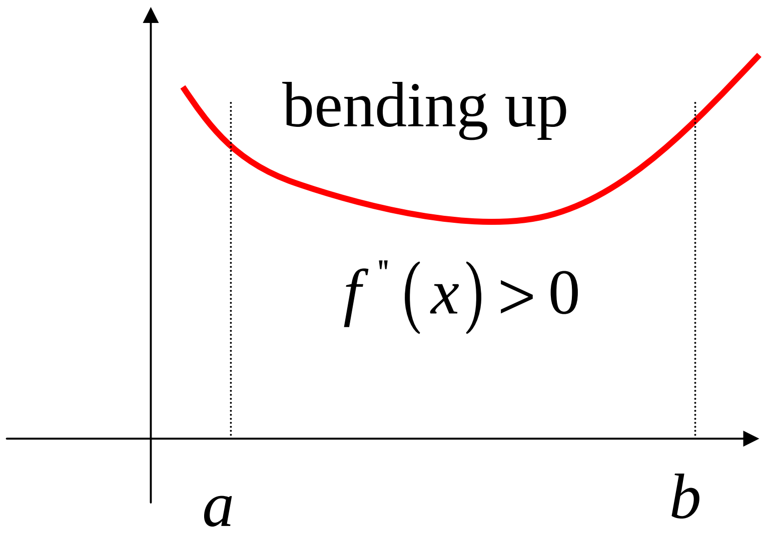
If $f^{'}(x) < 0$ for $a < x < b$

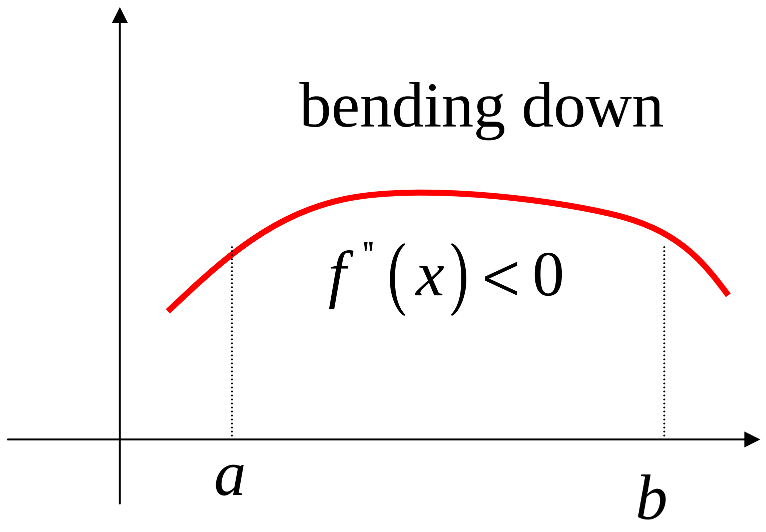
then $f(x)$ is decreasing on $[a,b]$



Bending-up and bending-down curves

Let $f(x)$ be continuous and $f'(x)$ and $f''(x)$ exist in $[a,b]$. The second derivative $f''(x)$ is the rate of change of the slope of the curve $y = f(x)$. If it is positive in the interval, the curve is bending up. If the second derivative is negative, the curve is bending down.

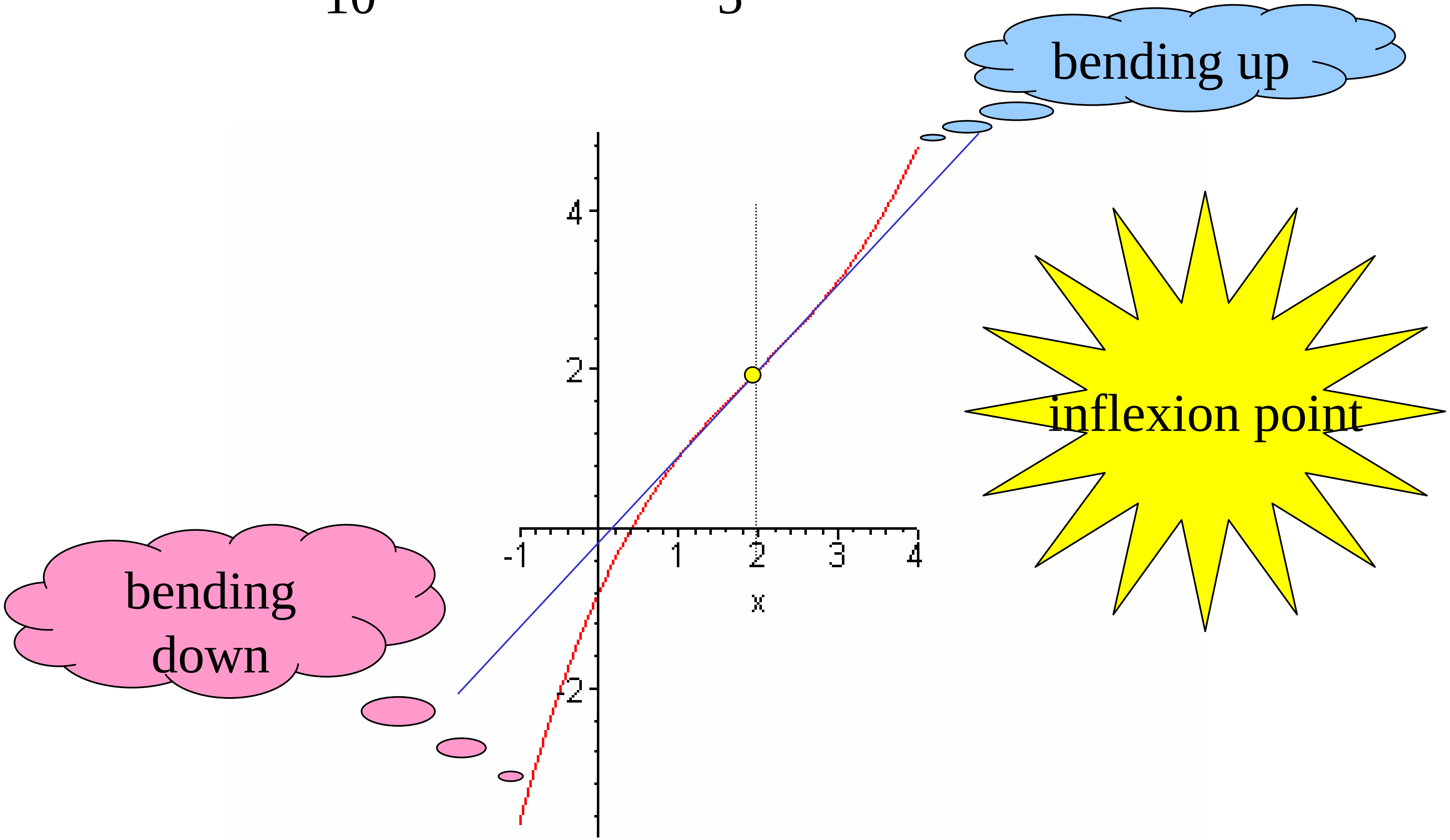




<u>Inflexion points</u>

A point at which a curve changes its behaviour from bending up to down (or vice versa) is called an inflexion point. If a curve is the graph of a function $f(x)$ whose second derivative exists and is continuous, then we must have $f''(x)=0$ at that point.

$$f(x)=\frac{(x-2)^3}{10}+x,\ f''(x)=\frac{3}{5}(x-2)\Rightarrow f''(2)=0$$



<u>Maximum and local maximum of a function</u>

Let a function $f(x)$ be given with a domain $D(f)$. For a $c \in D(f)$ we say that c is the point of a **<u>maximum</u>** of the function $f(x)$ if

$$\forall x \in D(f): f(c) \geq f(x)$$

If a neighbourhood $N(\delta) = (c - \delta, c + \delta)$ exists such that

$$\forall x \in N(\delta): f(c) \geq f(x)$$

we say that $f(x)$ has a **<u>local maximum</u>** at c.

Minimum and local minimum of a function

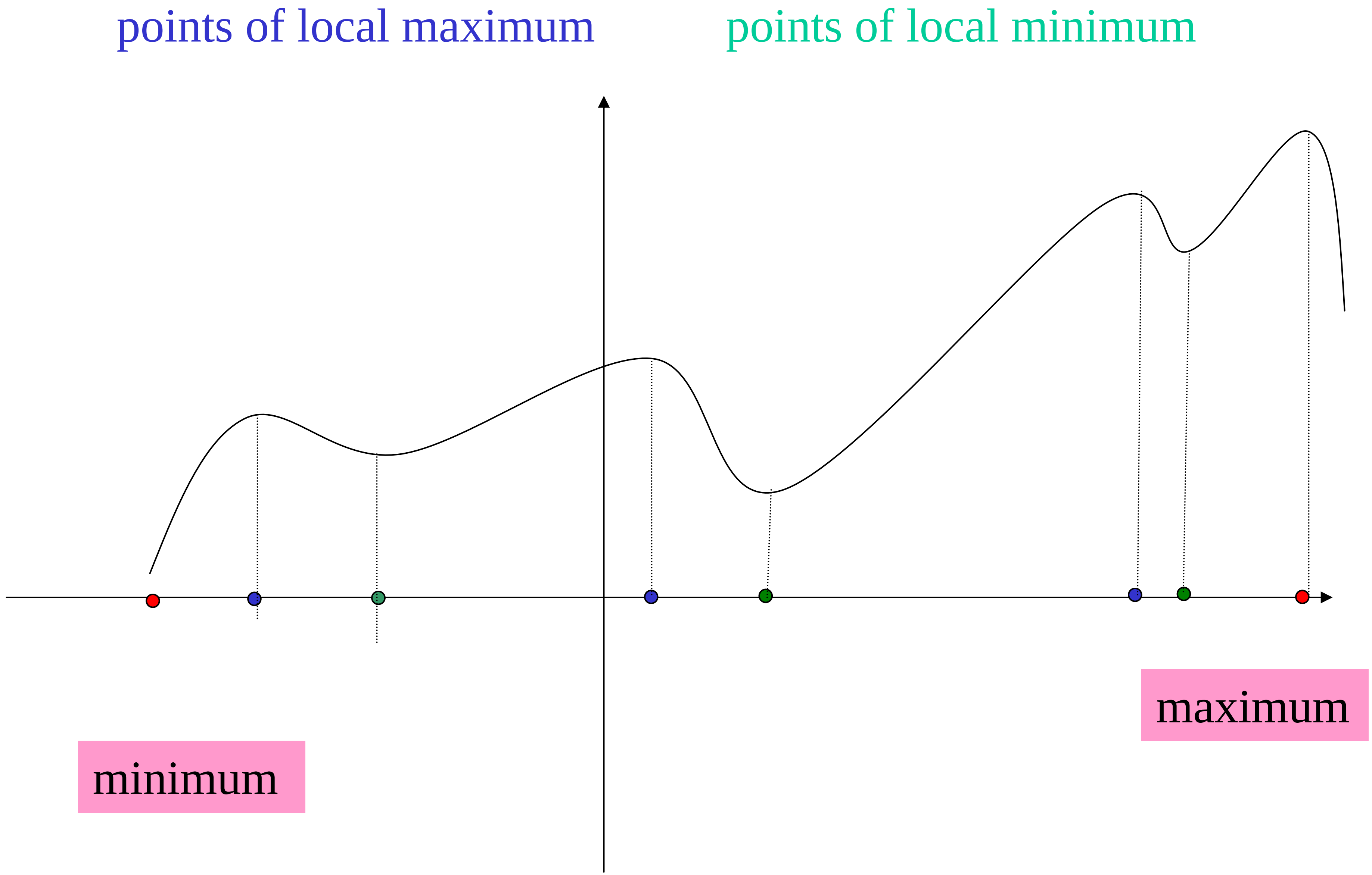
Let a function $f(x)$ be given with a domain $D(f)$. For a $c \in D(f)$ we say that c is the point of a **minimum** of the function $f(x)$ if

$$\forall x \in D(f) : f(c) \leq f(x)$$

If a neighbourhood $N(\delta) = (c - \delta, c + \delta)$ exists such that

$$\forall x \in N(\delta) : f(c) \leq f(x)$$

we say that $f(x)$ has a **local minimum** at c.

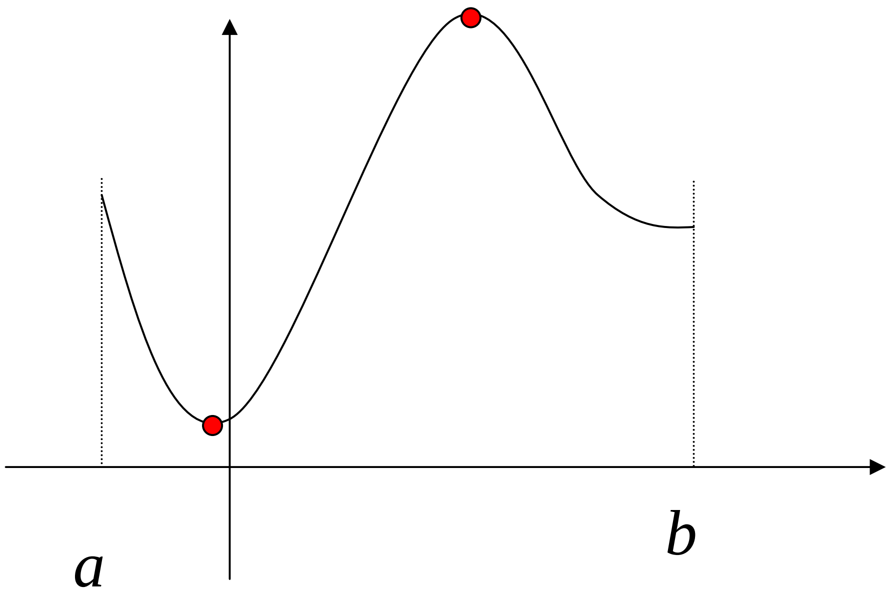

points of local maximum
points of local minimum
maximum
minimum

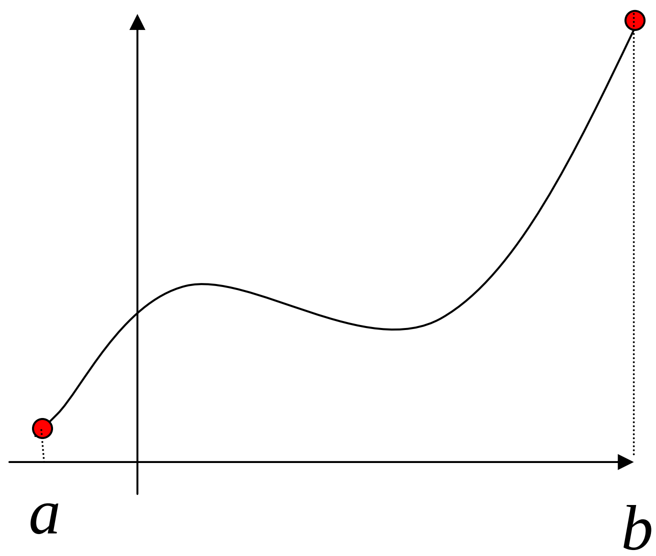
Let a continuous function $f(x)$ be defined on an interval $[a,b]$.

Let c be a point of maximum (minimum) of $f(x)$ on $[a,b]$.

Then $f(x)$ has a local maximum (minimum) at c

or c is one of the extreme points a and b





If a function $f(x)$ is to have a local maximum at a point c, then there must be a neighbourhood $(c-\delta, c)\cup(c, c+\delta)$ of c such that $f(x)$ is increasing in $(c-\delta, c)$ and decreasing in $(c, c+\delta)$

Thus $f^{'}(x)>0$ for $x\in(c-\delta, c)$ and $f^{'}(x)>0$ for $x\in(c, c+\delta)$

This is only possible if either $f^{'}(c)=0$ or the derivative of $f(x)$ does not exist at c.

For a local minimum, we may use a similar reasoning

A sufficient condition for a local minimum/maximum

If, for a function $f\left(x\right)$, we have

$$f^{\left(i\right)}\left(c\right)=0, i=1,2,\ldots,2k-1 \wedge f^{\left(2k\right)}\left(c\right)<0 \text{ for an integer } k>0$$

then $f\left(x\right)$ has a local maximum at c.

If

$$f^{\left(i\right)}\left(c\right)=0, i=1,2,\ldots,2k-1 \wedge f^{\left(2k\right)}\left(c\right)>0 \text{ for an integer } k>0$$

then $f\left(x\right)$ has a local minimum at c.

With k = 1 we have a particular case of this rule :

$f^{'}(c) = 0$ and $f^{''}(c) > 0$ for a minimum

and

$f^{'}(c) = 0$ and $f^{''}(c) < 0$ for a maximum

Example

Find minima and maxima of the function $y = 2x + 3\sqrt[3]{x^2}$

$$y' = 2 + 3\frac{2}{3}x^{-1/3} = 2 + \frac{2}{\sqrt[3]{x}} = \frac{2}{\sqrt[3]{x}}\left(\sqrt[3]{x} + 1\right)$$

$$\sqrt[3]{x} + 1 = 0 \Rightarrow x = -1$$

$$y'' = \frac{-2}{3}x^{-4/3} = \frac{-2}{3x\sqrt[3]{x}} \Rightarrow y''(-1) = \frac{-2}{3}$$

Maximum at -1.

Example

Verify if $f(x) = x^4$ has a local maximum or minimum

We have

$$f'(x) = 4x^3,\ f''(x) = 12x^2,\ f'''(x) = 24x,\ f^{(4)}(x) = 24$$

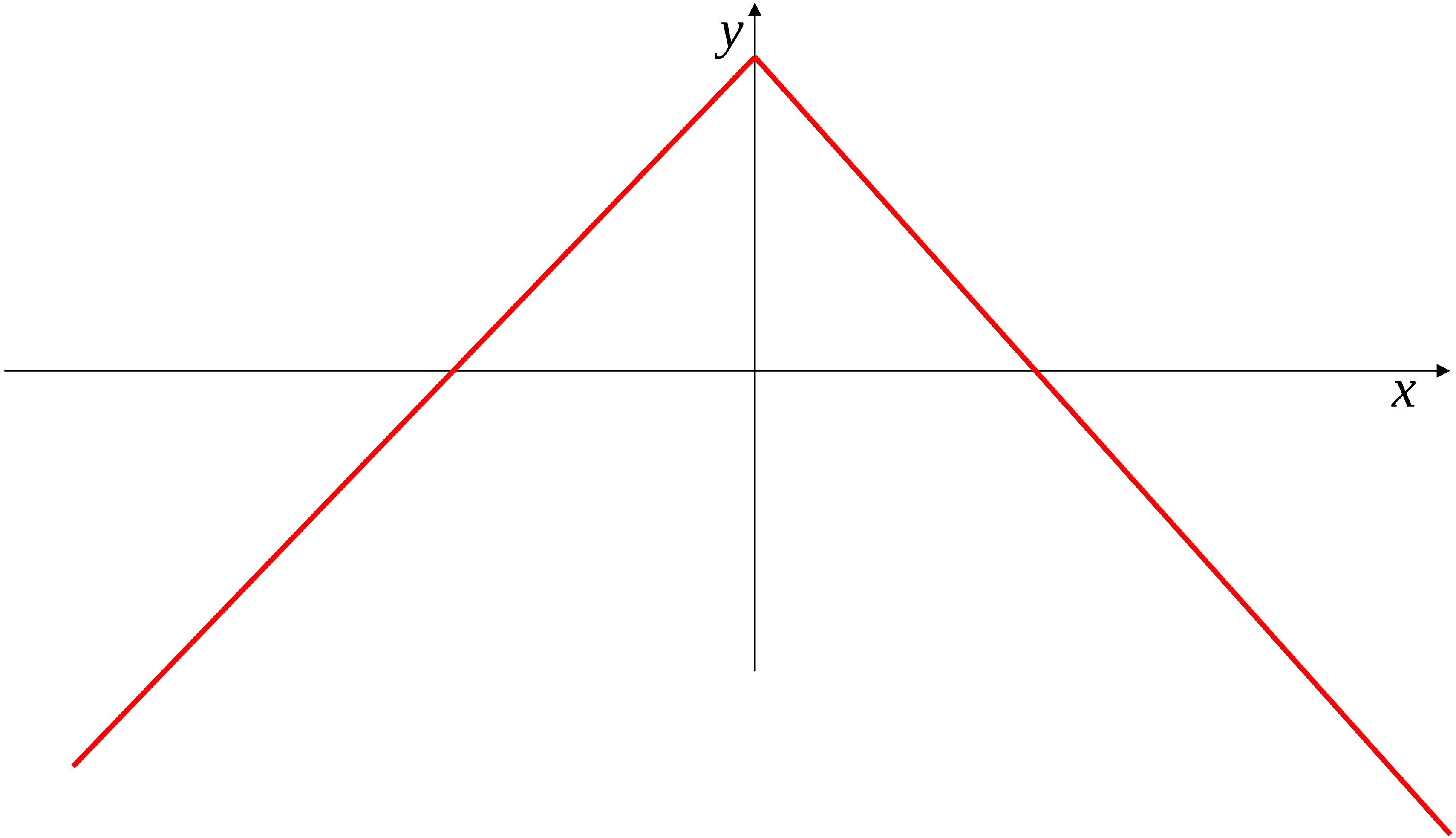
and so

$$f'(0) = f''(0) = f'''(0) = 0,\ f^{(4)}(0) > 0$$

We conclude that $f(x)$ has a local minimum at 0.

Example

Find the local minima and maxima of the function $y = 1 - |x|$



We have

$$y = \begin{cases} 1+x \text{ for } x<0 \\ \\ 1-x \text{ for } x\geq 0 \end{cases} \qquad y' = \begin{cases} 1 \text{ for } x<0 \\ \\ -1 \text{ for } x>0 \end{cases}$$

$$\lim_{x\to 0-} \frac{1+x-1}{x-0} = 1 \qquad \lim_{x\to 0+} \frac{1-x-1}{x-0} = -1$$

Thus at 0 $\lim_{x\to 0} \dfrac{1-|x|-(1-|0|)}{x-0}$ does not exist which means that

$y = 1-|x|$ has no derivative at 0. Since it is increasing on the left of 0 and decreasing on the right, we conclude that it has a local maximum at 0.

Find the maxima and minima of the function $y = x^4 - 2x^2$

on the interval [–2, 1.5]

$$y' = 4x^3 - 4x = 4x(x+1)(x-1)$$

Since local minima or maxima can only be reached at points where the first derivative is zero and minima or maxima can also be reached at the extreme points, the only candidates for maxima and minima are the points –2,–1,0,1,1.5. Calculating the values at these points we can verify that the minima are reached at the points –1 and 1 while the maximum is reached at the point –2