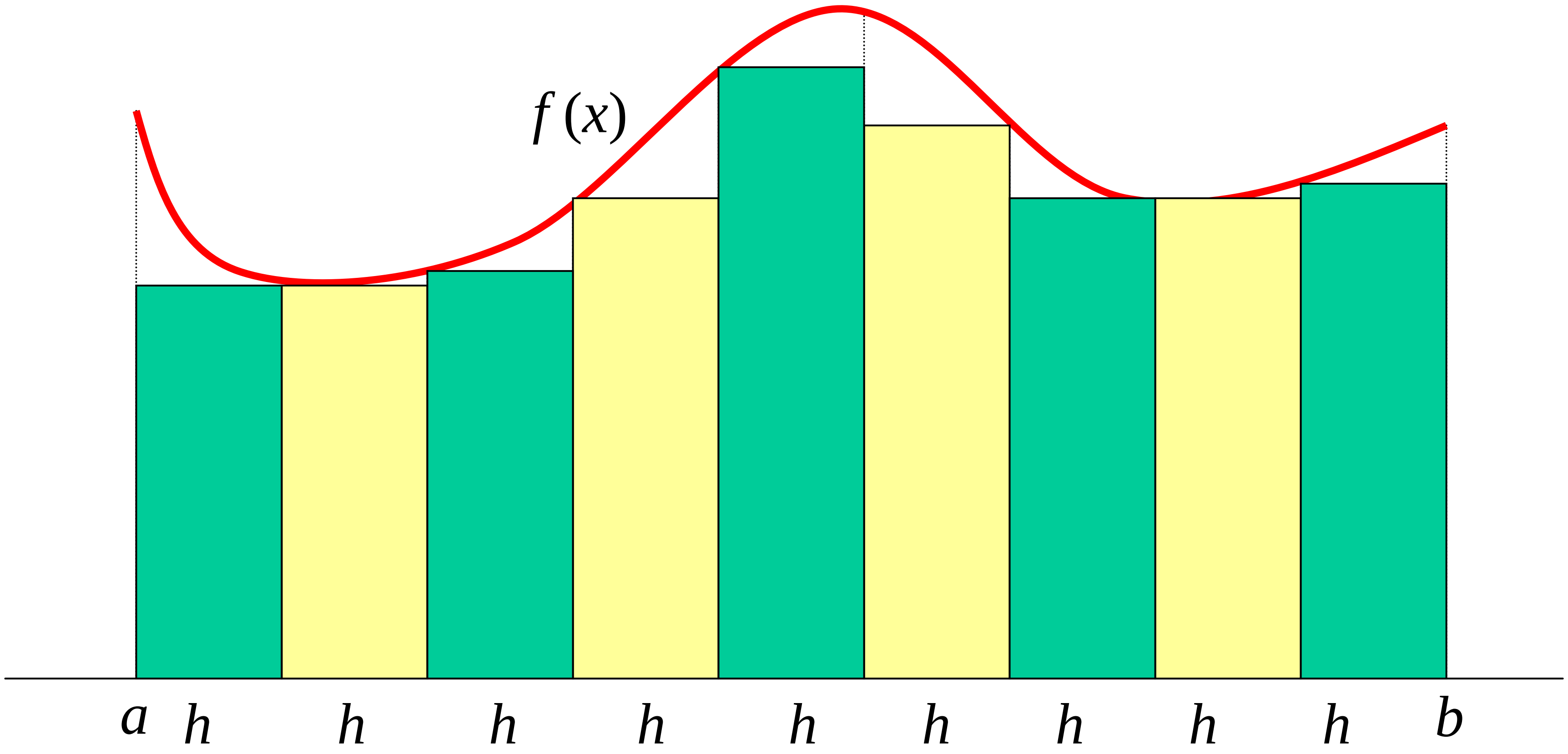
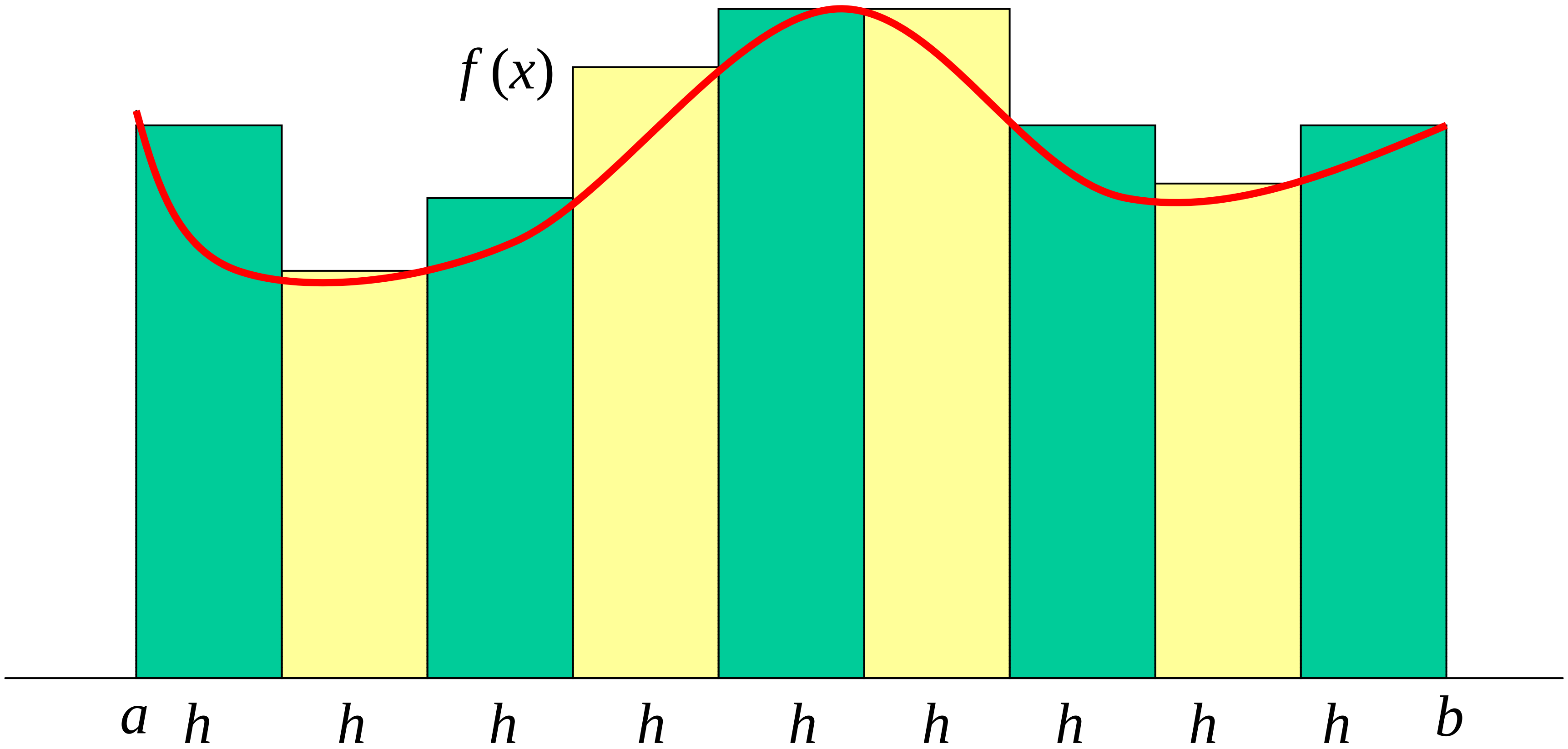
Lower sum

$$L(f,n,a,b)=\frac{b-a}{n}\sum_{i=1}^{n} f(\underline{x}_i),\ \underline{x}_i=\operatorname*{glb}_{x\in[x_{i-1},x_i)} f(x)$$

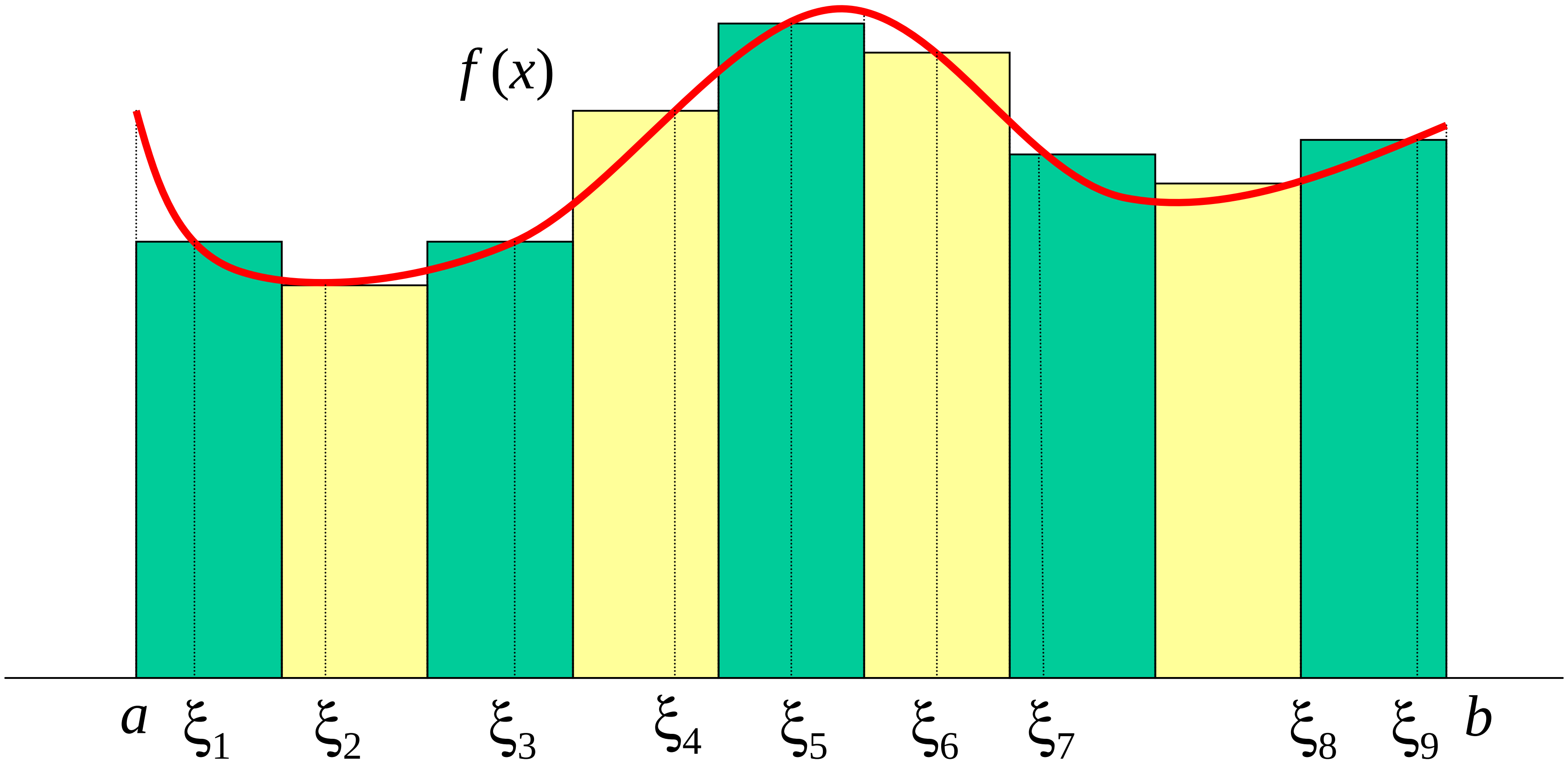


Upper sum $$U(f,n,a,b)=\frac{b-a}{n}\sum_{i=1}^{n} f\left(\bar{x}_i\right),\ \bar{x}_i=\operatorname*{lub}_{x\in[x_{i-1},x_i)} f(x)$$



Rieman sum

$$R(f,n,a,b)=\frac{b-a}{n}\sum_{i=1}^{n} f(\xi_i),\ \xi_i=[x_{i-1},x_i)$$



Lower integral

If the limit for $n \to \infty$ of the lower sum of a function $f(x)$ exists, we call it the lower integral of $f(x)$ over the interval $[a,b]$.

$$I_L\left(f,a,b\right)=\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n} f\left(\underline{x}_i\right) \quad \text{with} \quad \underline{x}_i = \underset{x\in\left[x_{i-1},x_i\right)}{\text{glb}} f\left(x\right)$$

Upper integral

If the limit for $n \to \infty$ of the upper sum of a function $f(x)$ exists, we call it the upper integral of $f(x)$ over the interval $[a,b]$.

$$I_U(f,a,b) = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(\underline{x}_i) \quad \text{with} \quad \underline{x}_i = \operatorname*{lub}_{x\in[x_{i-1},x_i)} f(x)$$

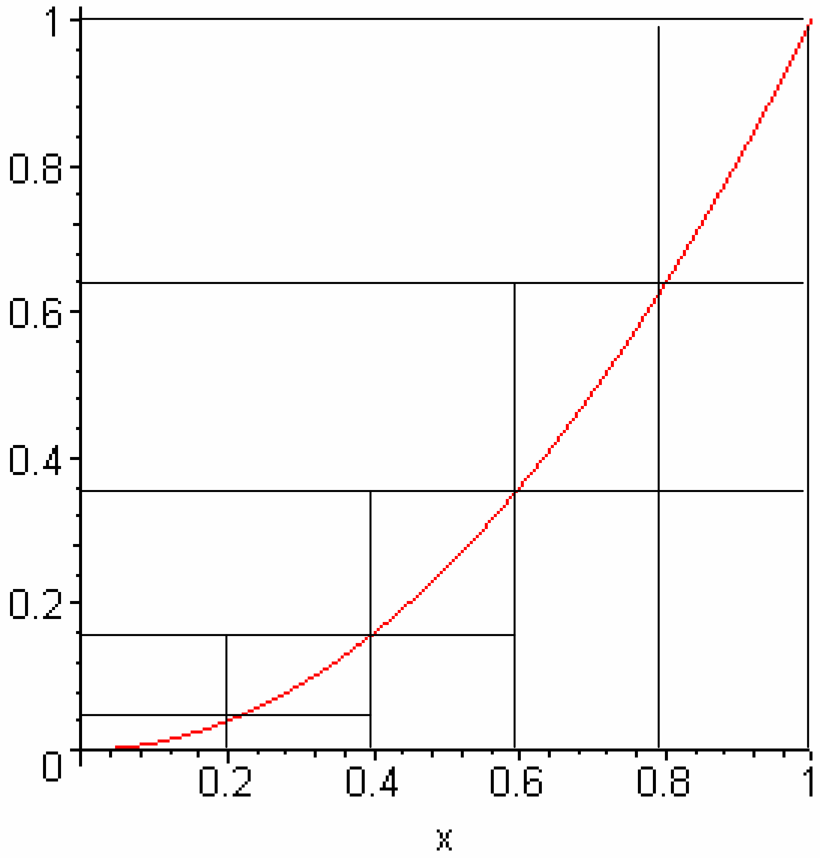
Rieman integral

If, for a function $f(x)$, both its lower and upper integrals exist and are the same, we say that $f(x)$ is Rieman-integrable and call the common value of the lower and upper integral the Rieman integral of $f(x)$. Formally

$$\boxed{I_L(f,a,b) = I_U(f,a,b) = \int_a^b f(x)\,dx}$$

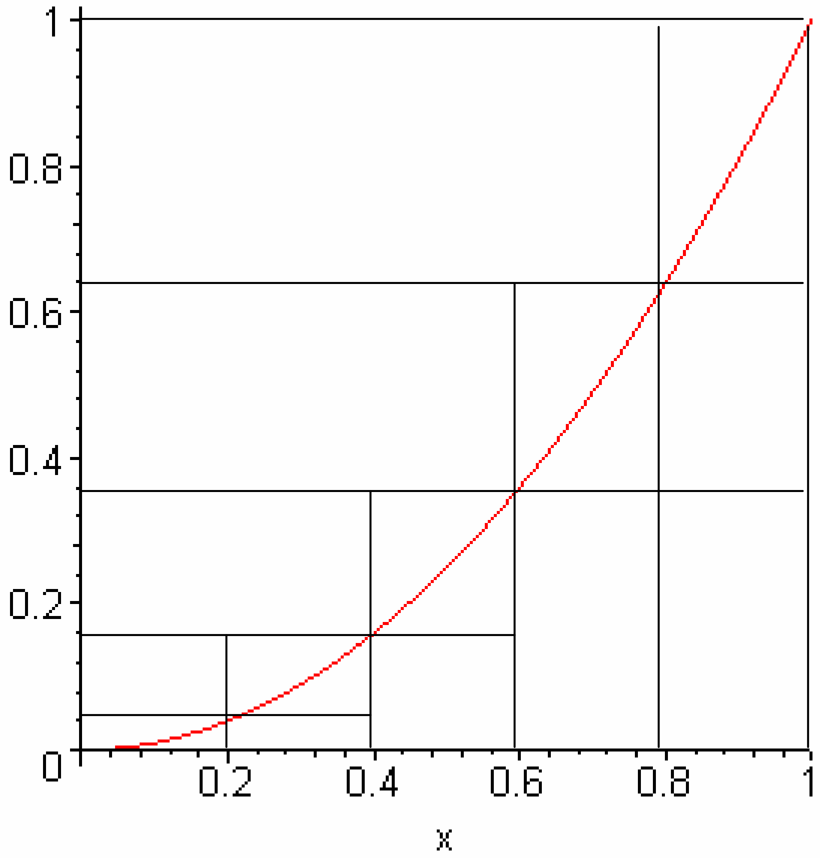
Example

Find the lower and upper integrals of the function $f(x) = x^2$ over the interval [0,1].



$$L\left(x^2,n,0,1\right)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2=\frac{1}{n^3}\sum_{i=0}^{n-1}i^2$$

$$U\left(x^2,n,0,1\right)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2=\frac{1}{n^3}\sum_{i=1}^{n}i^2$$



How can we calculate $S_n^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$?

Put $S_n^1 = 1 + 2 + 3 + \cdots + n = \dfrac{(n+1)n}{2}$

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1$$

$$\begin{aligned}
2^3 - 1^3 &= 3\cdot 1^2 + 3\cdot 1 + 1 \\
3^3 - 2^3 &= 3\cdot 2^2 + 3\cdot 2 + 1 \\
4^3 - 3^3 &= 3\cdot 3^2 + 3\cdot 3 + 1 \\
\vdots \quad \vdots \; & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \; \vdots \\
(n+1)^3 - n^3 &= 3n^2 + 3n + 1
\end{aligned}$$

$$(n+1)^3 - 1 = 3S_n^2 + 3S_n^1 + n$$

$$S_n^2 = \frac{1}{3}\left((n+1)^3 - 1 - 3S_n^1 - n\right)$$

$$S_n^2 = \frac{n}{6}\left(2n^2 + 3n + 1\right)$$

$$L\left(x^2,n,0,1\right)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2=\frac{1}{n^3}\sum_{i=0}^{n-1}i^2$$

$$L\left(x^2,n,0,1\right)=\lim_{n\to\infty}\frac{1}{n^3}\left(0+\frac{n}{6}\left(2n^2+3n+1\right)-n\right)$$

$$L\left(x^2,n,0,1\right)=\lim_{n\to\infty}\frac{2n^3+3n^2-5n}{6n^3}=\frac{1}{3}$$

$$U\left(x^2,n,0,1\right)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2=\frac{1}{n^3}\sum_{i=1}^{n}i^2$$

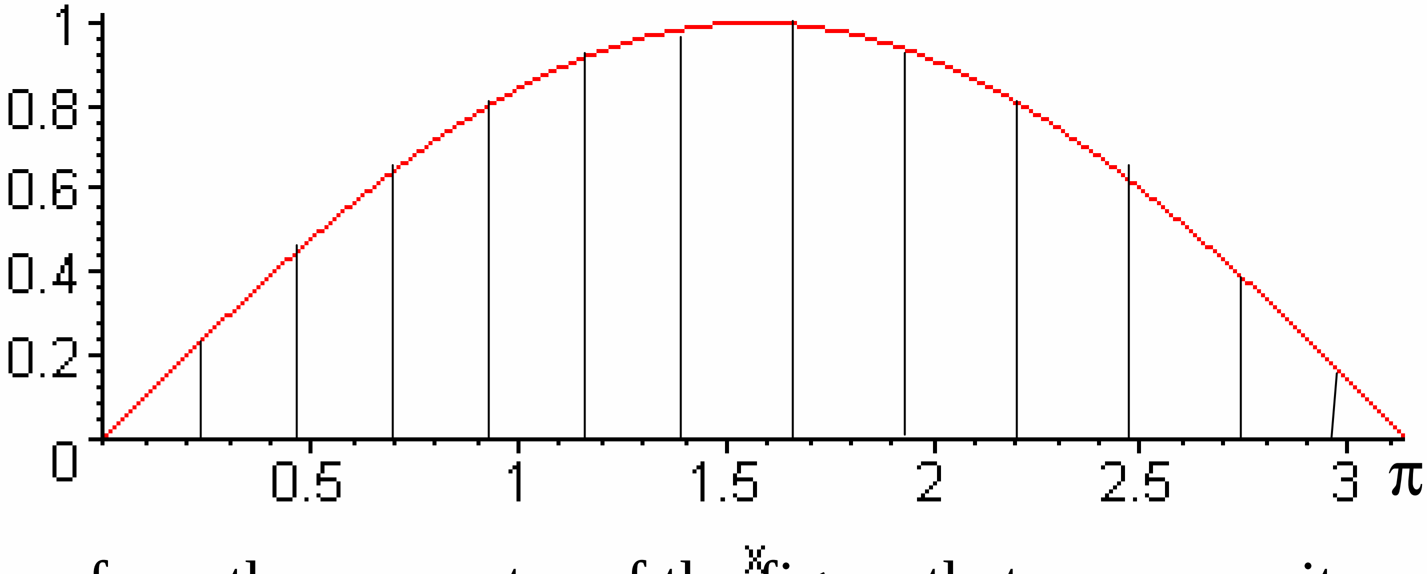
$$U\left(x^2,n,0,1\right)=\lim_{n\to\infty}\frac{1}{n^3}\left(\frac{n}{6}\left(2n^2+3n+1\right)\right)$$

$$U\left(x^2,n,0,1\right)=\lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2}=\frac{1}{3}$$

Thus we have calculated that $\int_0^1 x^2dx=\frac{1}{3}$

Example

Calculate $\int_0^{\pi} \sin x \, dx$



If follows from the symmetry of the figure that we can write

$$\int_0^{\pi} \sin x \, dx = 2 \int_0^{\pi/2} \sin x \, dx$$

Partition $\left[0,\frac{\pi}{2}\right]$ into n subintervals each having a length of $\frac{\pi}{2n}$

Set up $L\left(\sin x, n, 0, \frac{\pi}{2}\right) = \sum_{k=0}^{n-1} \frac{\pi}{2n} \sin\frac{k\pi}{2n} = \frac{\pi}{2n} \sum_{k=0}^{n-1} \sin\frac{k\pi}{2n}$

To calculate $\sum_{k=0}^{n-1} \sin\frac{k\pi}{2n}$ first note that

$$\cos\frac{k\pi}{2n} + i\sin\frac{k\pi}{2n} = e^{i\frac{k\pi}{2n}} = \left(e^{i\frac{\pi}{2n}}\right)^k$$

with $\left|e^{i\frac{\pi}{2n}}\right| \leq 1$

Clearly, $1, e^{i\frac{\pi}{2n}}, \left(e^{i\frac{\pi}{2n}}\right)^2, \left(e^{i\frac{\pi}{2n}}\right)^2, \left(e^{i\frac{\pi}{2n}}\right)^2, \ldots, \left(e^{i\frac{\pi}{2n}}\right)^{n-1}$

is a geometric sequence, which we can sum up:

$$1 + e^{i\frac{\pi}{2n}} + \left(e^{i\frac{\pi}{2n}}\right)^2 + \left(e^{i\frac{\pi}{2n}}\right)^2 + \left(e^{i\frac{\pi}{2n}}\right)^2 + \ldots + \left(e^{i\frac{\pi}{2n}}\right)^{n-1} = \frac{1 - \left(e^{i\frac{\pi}{2n}}\right)^n}{1 - e^{i\frac{\pi}{2n}}}$$

Let us now calculate the limit

$$\lim_{n\to\infty}\frac{\pi}{2n}\frac{1-\left(e^{i\frac{\pi}{2n}}\right)^n}{1-e^{i\frac{\pi}{2n}}}=\frac{\pi}{2n}\lim_{n\to\infty}\frac{1-e^{i\frac{\pi}{2}}}{1-e^{i\frac{\pi}{2n}}}=\frac{\pi}{2n}\lim_{n\to\infty}\frac{1-\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)}{1-e^{i\frac{\pi}{2n}}}=$$

$$=\frac{\pi}{2n}\lim_{n\to\infty}\frac{1-i}{1-e^{i\frac{\pi}{2n}}}=\pi(1-i)\lim_{n\to\infty}\frac{\frac{1}{2n}}{1-e^{i\frac{\pi}{2n}}}=\pi(1-i)\lim_{n\to\infty}\frac{\frac{-1}{2n^2}}{\frac{i\pi}{2n^2}e^{i\frac{\pi}{2n}}}=$$

$$=\pi(1-i)\frac{-1}{i\pi}=\frac{i-1}{i}=\frac{i\cdot i-i}{i\cdot i}=\frac{-1-i}{-1}=1+i$$

Thus we have established that

$$1+i=\lim_{n\to\infty}\frac{\pi}{2n}\sum_{k=0}^{n-1}e^{i\frac{k\pi}{2n}}=\lim_{n\to\infty}\frac{\pi}{2n}\sum_{k=0}^{n-1}\left(\cos\frac{k\pi}{2n}+i\sin\frac{k\pi}{2n}\right)=$$

$$=\lim_{n\to\infty}\frac{\pi}{2n}\sum_{k=0}^{n-1}\cos\frac{k\pi}{2n}+i\lim_{n\to\infty}\frac{\pi}{2n}\sum_{k=0}^{n-1}\sin\frac{k\pi}{2n}=\int_0^{\pi/2}\cos x\,dx+i\int_0^{\pi/2}\sin x\,dx$$

This means that $\int_0^{\pi/2}\cos x\,dx=\int_0^{\pi/2}\sin x\,dx=1$

and so $\int_0^{\pi}\sin x\,dx=2$

The previous examples have shown that finding the Rieman integral of even relatively simple functions by using only the definition of the Rieman integral may be a very tedious job. Fortunately, there are other methods, too.

Theorem

If a function $f(x)$ is integrable over $[a,b]$, then the limit of its Rieman sum is equal to the integral of $f(x)$ over $[a,b]$.

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(\xi_i) = \int_a^b f(x)\, dx$$

To prove this, it is suficient to realize that

$L(f,n,a,b) \le R(f,n,a,b) \le U(f,n,a,b)$ for every n, since

$$\operatorname*{glb}_{x\in[x_{i-1},x_i)} f(x) \le f(\xi_i) \le \operatorname*{lub}_{x\in[x_{i-1},x_i)} f(x) \quad \text{for any} \quad \xi_i \in [x_{i-1}, x_i)$$

The rest then follows from the squeezing rule.

Let an integrable function f (x) be defined on an interval [a,b] and let it have an antiderivative F (x), that is,

$$F(x) = \int f(x)\,dx$$

Let us divide [a,b] into n equal subintervals, denoting by x_i the dividing points and putting $a = x_0$, $b = x_n$

then we can write

$$F(b) - F(a) = \sum_{i=1}^{n}\left[F(x_i) - F(x_{i-1})\right] = \sum_{i=1}^{n}\left[(x_i - x_{i-1})\, f(\xi_i)\right]$$

where $\xi_i \in (x_{i-1}, x_i), i = 1, 2, \ldots, n$ using the mean value theorem.

$$F(b)-F(a)=\sum_{i=1}^{n}\left[F(x_i)-F(x_{i-1})\right]=\sum_{i=1}^{n}\left[(x_i-x_{i-1})f(\xi_i)\right]$$

holds for all integers n and thus it could be easily proved that this equality will also hold if we take the limit for $n\to\infty$

Then $F(b)-F(a)=\lim_{n\to\infty}\sum_{i=1}^{n}\left[(x_i-x_{i-1})f(\xi_i)\right]$ and since $f(x)$ is integrable over $[a,b]$, we have

$$\boxed{F(b)-F(a)=\int_a^b f(x)\,dx}$$

Newton-Leibnitz formula