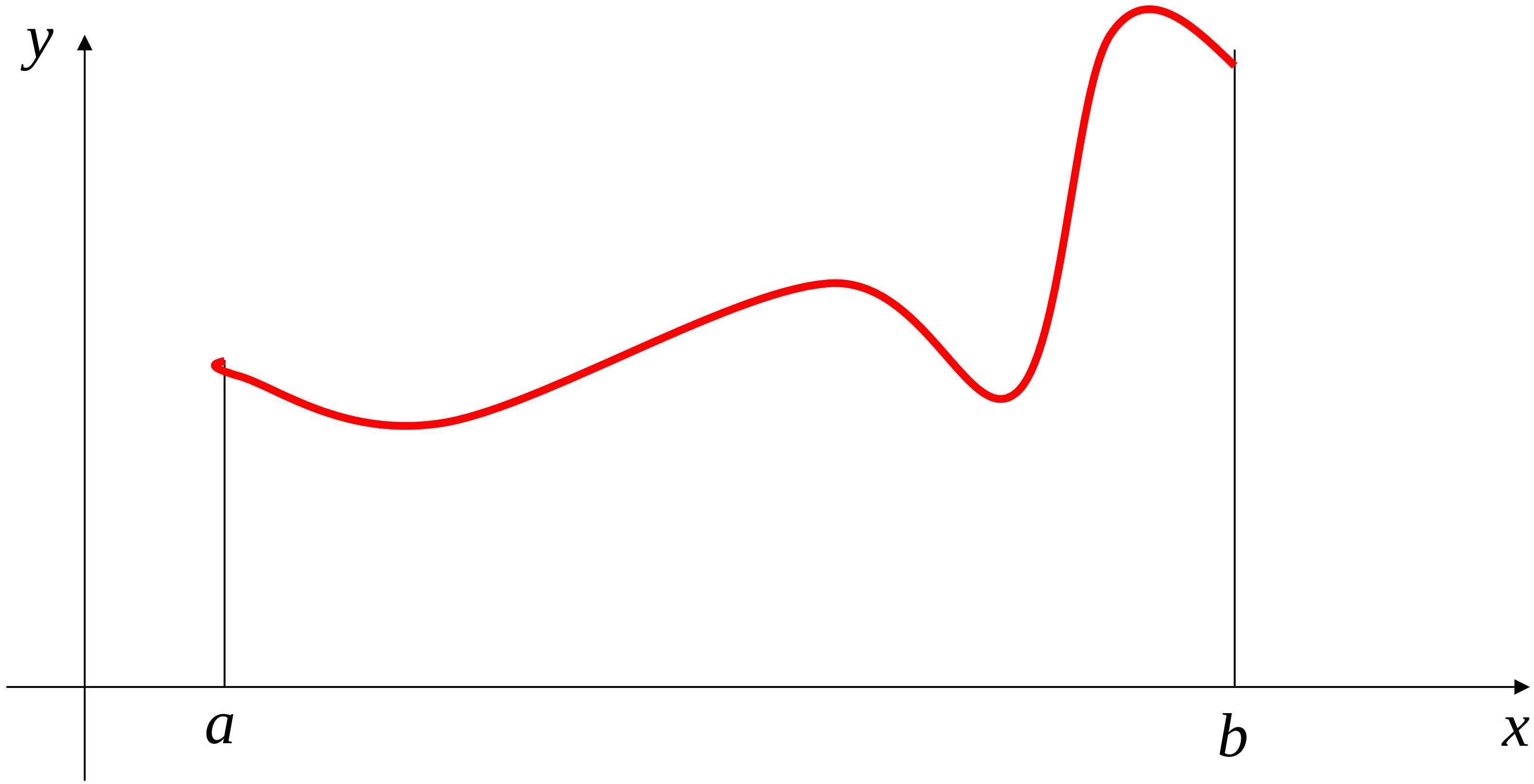
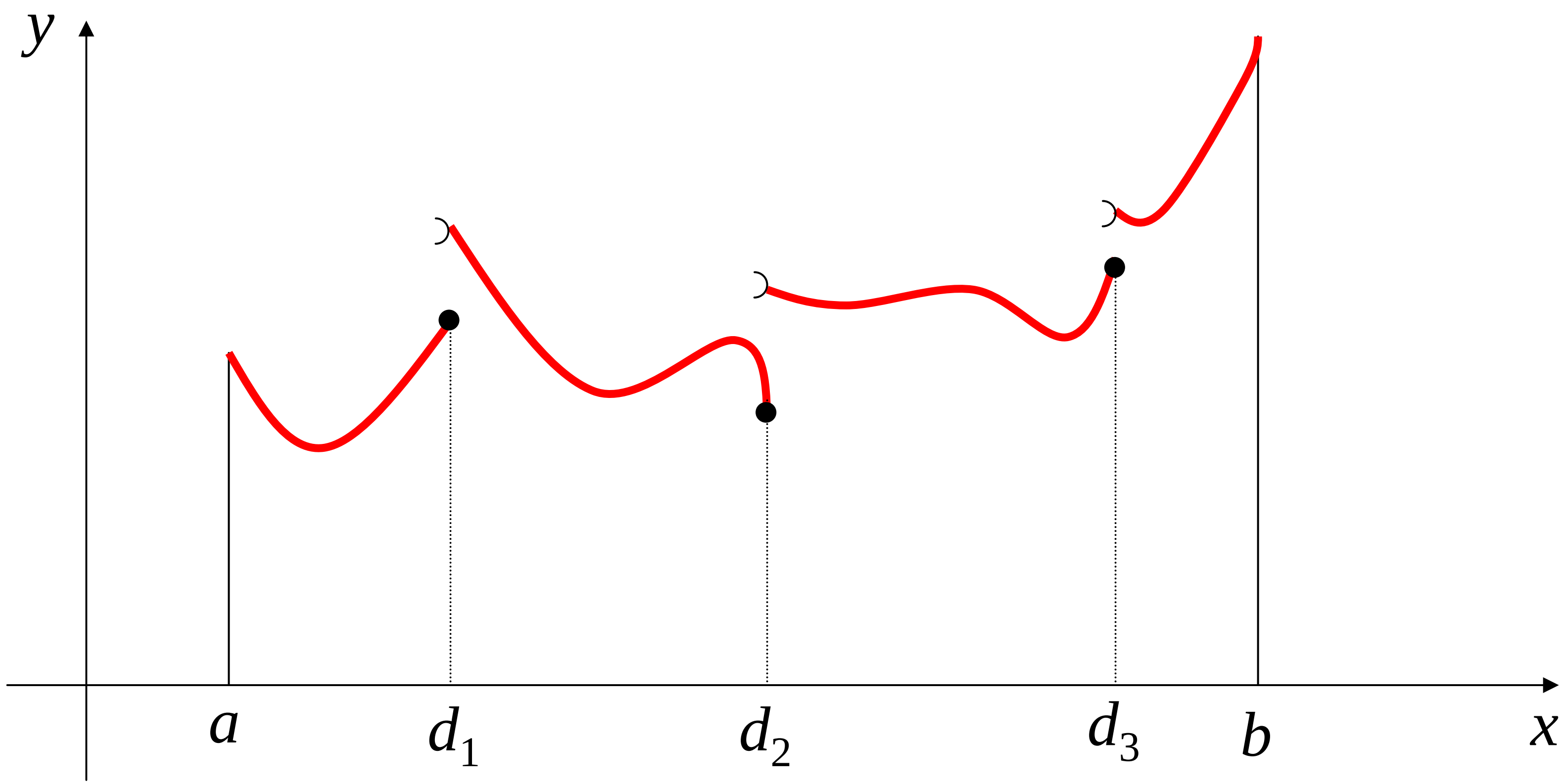
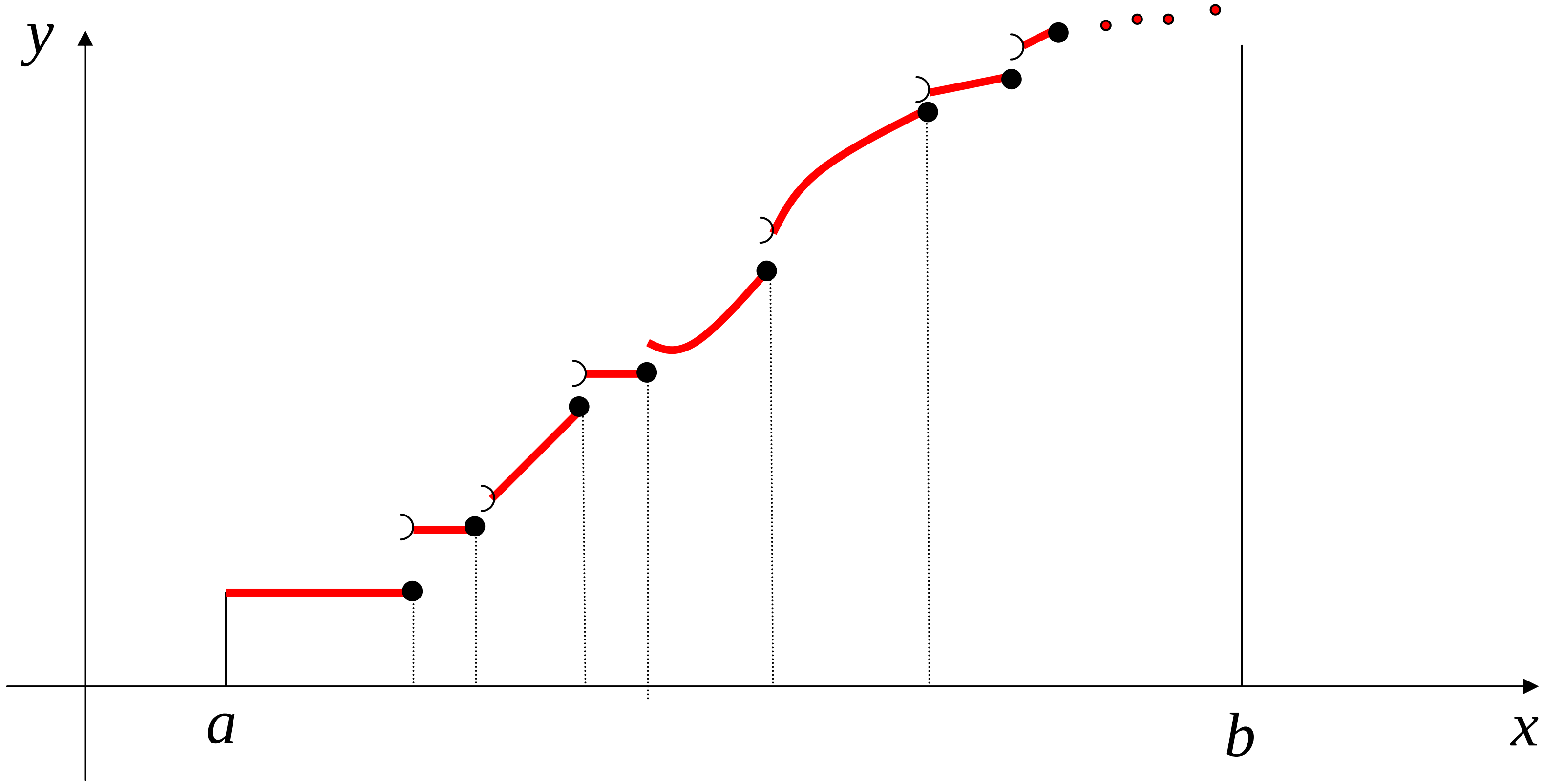
If a function $f(x)$ is continuous on an interval $[a,b]$, then it is Rieman-integrable.



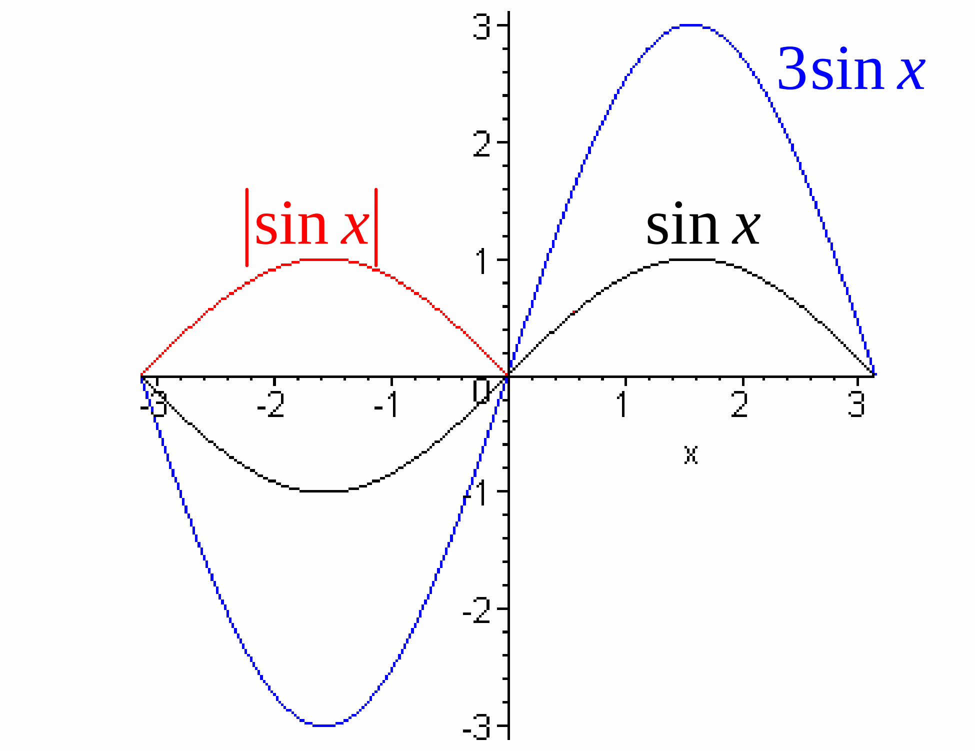
If a function $f(x)$ is bounded on an interval $[a,b]$, and has at most a finite number of points of discontinuity, then it is Rieman-integrable.



If a function $f(x)$ is monotonous and bounded on an interval $[a,b]$, then it is Rieman-integrable.

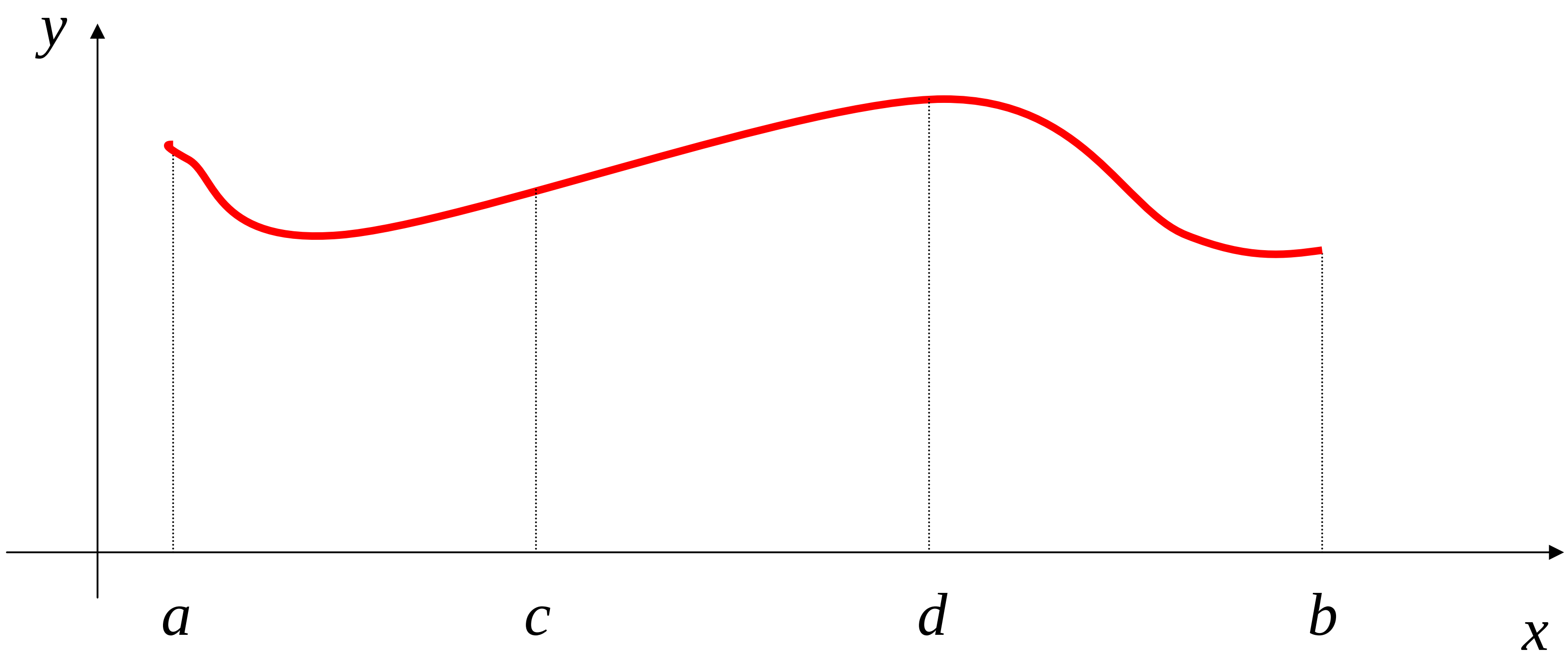


If a function $f(x)$ is integrable on an interval $[a,b]$, then so are functions $|f(x)|$ and $kf(x)$ where $k = \text{const}$.

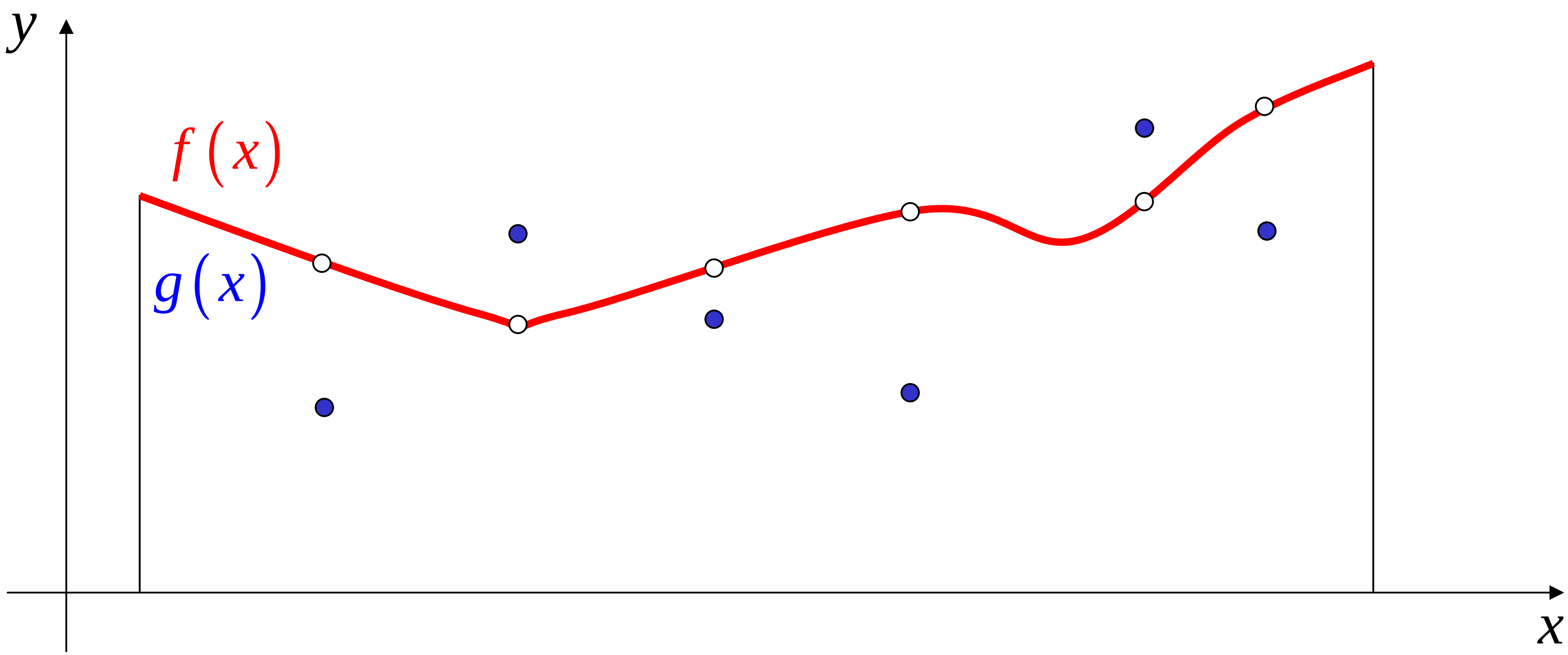


If the functions $f(x), g(x)$ are integrable on an interval $[a,b]$, so are the functions $f(x)+g(x)$, $f(x)-g(x)$, and $f(x)\cdot g(x)$

If a function $f(x)$ is integrable on an interval $[a,b]$, then it is also integrable on any subinterval $[c,d] \subseteq [a,b]$. If, in turn, the interval $[a,b]$ is broken down into parts and $f(x)$ is integrable on each of the parts separately, then $f(x)$ is integrable on $[a,b]$.



If a function $f(x)$ is integrable on an interval $[a,b]$, then a function $g(x)$ such that $f(x)=g(x)$, for every $x\in[a,b]$ except a finite number of points $x_1,x_2,\ldots,x_k\in[a,b]$ at which $f(x)\neq g(x)$, then $g(x)$ is also integrable on $[a,b]$.



A function may be integrable even if it does not satisfy any of the above conditions as illustrated by the following example:

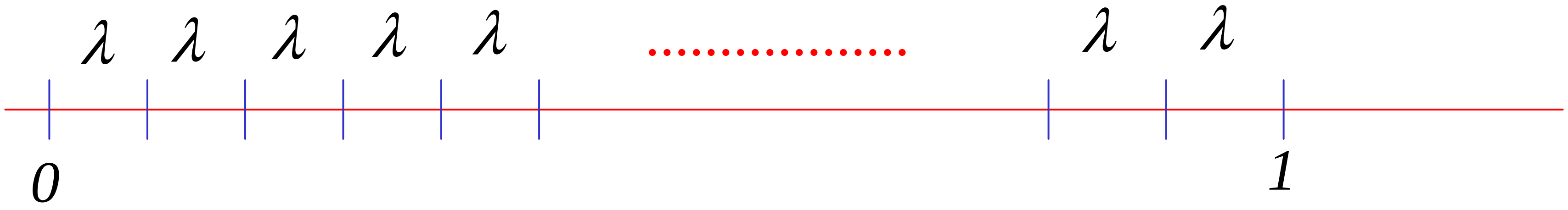
Let $f(x)$ be a function defined on [0,1] as follows

$$f(x) = \begin{cases} \dfrac{1}{q} \text{ if } x = \dfrac{p}{q} \text{ where } p, q \text{ are relatively prime integers, } q > 0 \\ 0 \text{ otherwise} \end{cases}$$

We shall prove that it is integrable on [0,1].

Let $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1}, x_n = 1$ be a partition of [0,1] such that $x_i - x_{i-1} \le \lambda, i = 1, 2, \ldots, n.$ Thus $\lambda = \dfrac{b-a}{n}$

For any natural number N, we can divide the partition intervals into two groups.



Intervals of the first group will be those that contain a number $\frac{p}{q}$ such that $q \le N$ and p, q are relatively prime integers. Since $\frac{p}{q} \le 1$, we have $p \le q$, which means that there is only a finite number k of such numbers. Since, in the extreme cases, every such number could lie at the border of two partition intervals, there are at most $2k$ such intervals. Their total length L_1 then does not exceed $2\lambda k$. To denote that the number k depends on N, we will write $L_1 = 2\lambda k(N)$

Intervals of the second group will only contain rational numbers $\frac{p}{q} \leq 1$ such that $q > N$ so that, for any x in an interval of the second type, we have $f(x) < \frac{1}{N}$

If we denote by M_i, m_i, the minimum and maximum of $f(x)$ on $[x_{i-1}, x_i]$ respectively, then $M_i - m_i < \frac{1}{N}$ if $[x_{i-1}, x_i]$ is a partition interval of the second type.

To prove that $f(x)$ is integrable on [0,1] we have now to show that, for every $\varepsilon > 0$, we can choose λ such that

$$\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) < \varepsilon$$

However, we can write

$$\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = \sum(M_j - m_j)(x_j - x_{j-1}) + \sum(M_r - m_r)(x_r - x_{r-1})$$

where the first summation is performed over all the intervals of the first group and the second over the intervals of the second group.

From what was said above, we can conclude that

$$\sum\left(M_j - m_j\right)\left(x_j - x_{j-1}\right) + \sum\left(M_r - m_r\right)\left(x_r - x_{r-1}\right) \le 2k\left(N\right)\lambda + \frac{1}{N}$$

Now if we first choose $N > \frac{2}{\varepsilon}$ and then $\lambda < \frac{\varepsilon}{4k\left(N\right)}$, we get

$$\sum_{i=1}^{n}\left(M_i - m_i\right)\left(x_i - x_{i-1}\right) < 2k\left(N\right)\frac{\varepsilon}{4k\left(N\right)} + \frac{1}{\frac{2}{\varepsilon}} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

In the following example we will define a function that is not integrable over the interval [0,1].

$$\chi(x) = \begin{cases} 1 \text{ if } x \text{ is a rational number} \\ \\ 0 \text{ if } x \text{ is not a rational number} \end{cases}$$

This is the so-called Dirichlet function. Clearly, for every partition

$a = x_0 < x_1 < x_2 < \cdots < x_{n-1}, x_n = b$ of [0,1], each subinterval $\left[x_{i-1}, x_i\right]$

contains both rational and irrational numbers so that $M_i - m_i = 1$

and $\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^{n}(x_i - x_{i-1}) = b - a$, which means

that $\chi(x)$ is not integrable over [0,1].

There are other ways of defining a definite integral. An example might be the Lebesgue integral.

Functions that are Rieman integrable are also Lebesgue-integrable but there are also functions, such as the Dirichlet function, that are not Rieman integrable but are Lebesgue-integrable.