If $F(x)$ is a function such that $\dfrac{dF(x)}{dx} = f(x)$

we write $F(x) = \int f(x)\,dx$ and say that $F(x)$ is an integral of $f(x)$

Since the derivative of a constant function is zero, we usually write

$$\int f(x)\,dx = F(x) + c$$

DIFFERENTIATION

$$x^n \leftrightarrow nx^{n-1}, n \neq 0$$

$$e^x \leftrightarrow e^x$$

$$\ln x \leftrightarrow 1/x$$

$$\sin x \leftrightarrow \cos x$$

$$\cos x \leftrightarrow -\sin x$$

$$\tan x \to 1/\cos^2 x$$

$$\cot x \leftrightarrow -1/\sin^2 x$$

$$\arcsin x \leftrightarrow 1/\sqrt{1-x^2}$$

$$\arccos x \leftrightarrow -1/\sqrt{1-x^2}$$

$$\arctan x \leftrightarrow 1/\left(1+x^2\right)$$

$$\operatorname{arccot} x \leftrightarrow -1/\left(1+x^2\right)$$

$$\sinh x \leftrightarrow \cosh x$$

$$\cosh x \leftrightarrow \sinh x$$

$$\tanh x \leftrightarrow 1/\cosh^2 x$$

$$\coth x \leftrightarrow -1/\sinh^2 x$$

$$\operatorname{arcsinh} x \leftrightarrow 1/\sqrt{x^2+1}$$

$$\operatorname{arccosh} x \leftrightarrow 1/\sqrt{x^2-1}, x \in (1,\infty)$$

$$\operatorname{arctanh} x \leftrightarrow 1/1-x^2, x \in (-1,1)$$

$$\operatorname{arccoth} x \leftrightarrow 1/1-x^2, x \in R-(-1,1)$$

INTEGRATION

Since

$$\operatorname{arcsinh} x = \ln\left(x+\sqrt{x^2+1}\right) \qquad \operatorname{arccosh} x = \ln\left(x+\sqrt{x^2-1}\right)$$

$$\operatorname{arctanh} x = \operatorname{arccoth} x = \ln\sqrt{\left|\frac{1+x}{1-x}\right|}$$

we have $$\ln\left(x+\sqrt{x^2+1}\right) \quad \leftrightarrow \quad 1/\sqrt{x^2+1}$$

$$\ln\left(x+\sqrt{x^2-1}\right) \quad \leftrightarrow \quad 1/\sqrt{x^2-1}\,,\, x\in(1,\infty)$$

$$\ln\sqrt{\left|\frac{1+x}{1-x}\right|} \quad \leftrightarrow \quad 1/1-x^2\,,\, x\in(-1,1)$$

$$\ln\sqrt{\left|\frac{1+x}{1-x}\right|} \quad \leftrightarrow \quad 1/1-x^2\,,\, x\in R-(-1,1)$$

Two easy rules

- $$\int \left(f(x) \pm g(x) \right) dx = \int f(x)\,dx \pm \int g(x)\,dx$$

- $$\int a\,f(x)\,dx = a \int f(x)\,dx$$

Integration by parts

$$\frac{d\left(f(x)g(x)\right)}{dx} = f(x)' g(x) + f(x) g(x)'$$

$$f(x)g(x) = \int f(x)' g(x)\,dx + \int f(x) g(x)'\,dx$$

$$\boxed{\int f(x)' g(x)\,dx = f(x)g(x) - \int f(x) g(x)'\,dx}$$

<u>Example</u>

$$\int x^2 \sin x \, dx = \left| \begin{matrix} f' = \sin x & f = -\cos x \\ g = x^2 & g' = 2x \end{matrix} \right| = -x^2 \cos x + 2\int x \cos x \, dx =$$

$$\int x \cos x \, dx = \left| \begin{matrix} f' = \cos x & f = \sin x \\ g = x & g' = 1 \end{matrix} \right| = x \sin x - \int \sin x \, dx = x \sin x + \cos x$$

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2\cos x + c$$

Example

$$\int \sin x e^x dx = \left| \begin{matrix} f' = \sin x & f = -\cos x \\ g = e^x & g' = e^x \end{matrix} \right| = -\cos x e^x + \int \cos x e^x dx =$$

$$\int \cos x e^x dx = \left| \begin{matrix} f' = \cos x & f = \sin x \\ g = e^x & g' = e^x \end{matrix} \right| = \sin x e^x - \int \sin x e^x dx =$$

$$\int \sin x e^x dx = -\cos x e^x + \sin x e^x - \int \sin x e^x dx$$

$$2\int \sin x e^x dx = -\cos x e^x + \sin x e^x$$

$$\int \sin x e^x dx = \frac{(-\cos x + \sin x) e^x}{2} + c$$

<u>Integration by substitution</u>

We are to calculate $\int f(x)\,dx$ and suppose that we manage to find a $u(x)$ such that $f(x) = g(u(x)).u^{'}(x)$. Then we can write

$$\int f(x)\,dx = \int g(u(x))\frac{du}{dx}dx = \int g(u)\,du$$

If, in turn, we can view the independent variable *x* as a function $x = \varphi(t)$ then

$$\int f(x)\,dx = \int f(\varphi(t))\frac{dx}{dt}dt = \int f(\varphi(t))\,\varphi^{'}dt$$

We can then use the following schemes for integration

$$\int f(x)\,dx = \left| \begin{array}{l} u(x) = u \\ u^{'}dx = du \end{array} \right| = \int g(u)\,du$$

$$\int f(x)\,dx = \left| \begin{array}{l} x = \varphi(t) \\ dx = \varphi^{'}dt \end{array} \right| = \int f(\varphi(t))\,\varphi^{'}dt$$

Example

$$\int \sin^2 x \cos x \, dx = \left| \begin{matrix} \sin x = u \\ \cos x dx = du \end{matrix} \right| = \int u^2 du = \frac{1}{3} u^3 + c =$$

$$= \frac{1}{3} \sin x^3 + c$$

Example

$$\int\sqrt{1-x^2}\,dx = \left|\begin{array}{c} x=\sin t \\ dx=\cos t\,dt \end{array}\right| = \int\sqrt{1-\sin^2 t}\,\cos t\,dt = \int\cos^2 t\;dt =$$

$$= \int\frac{1+\cos 2t}{2} = \frac{t}{2}+\frac{1}{2}\int\cos 2t\;dt$$

$$\int\cos 2t\;dt = \left|\begin{array}{c} 2t=u \\ 2dt=du \end{array}\right| = \frac{1}{2}\int\cos u\;du = \frac{\sin u}{2} = \frac{\sin 2t}{2} = \sin t\cos t$$

$$\int\sqrt{1-x^2}\,dx = \frac{t+\sin t\cos t}{2}+c \qquad t=\arcsin x$$

$$\int\sqrt{1-x^2}\,dx = \frac{\arcsin x + x\sqrt{1-x^2}}{2}+c$$