We want to calculate the integral

$$\int \frac{N_m(x)}{D_n(x)}\,dx = \int \frac{a_0 x^m + a_1 x^{m-1} + \cdots + a_{m-1}x + a_m}{b_0 x^n + b_1 x^{n-1} + \cdots + b_{n-1}x + b_n}\,dx$$

We can assume that $m < n$. If this is not the case, we can divide the numerator by the denominator using the remainder as a new numerator.

## Example

$$\frac{x^9+17x^8+109x^7+339x^6+614x^5+862x^4-831x^3+665x^2+325x+125}{x^7+17x^6+108x^5+322x^4+506x^3+540x^2+325x+125}$$

$$x^2+1+\frac{x^4-3x^3+2x-1}{x^7+17x^6+108x^5+322x^4+506x^3+540x^2+325x+125}$$

The denominator can be written as

$$D_n(x) = b_0 \left(x - \alpha_1\right)^{r_1} \cdots \left(x - \alpha_s\right)^{r_s} \left(x^2 + \beta_1 x + \gamma_1\right)^{q_1} \cdots \left(x^2 + \beta_t x + \gamma_t\right)^{q_t}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_s$ are the roots of $D(x)$ with multiplicities $r_1, r_2, \ldots, r_s$ and, for $i = 1, 2, \ldots, t$ , we have $\beta_i^2 - 4\gamma_i < 0$. Clearly,

$$\sum_{i=1}^{s} r_i + \sum_{j=1}^{t} q_j = n$$

It can be shown that

$$\frac{a_0 x^m + a_1 x^{m-1} + \cdots + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n}$$

can be written as

$$\sum_{i=1}^{r_1} \frac{A_{1i}}{\left(x-\alpha_1\right)^i} + \cdots + \sum_{i=1}^{r_s} \frac{A_{si}}{\left(x-\alpha_s\right)^i} + \sum_{i=1}^{q_1} \frac{B_{1i}x + C_{1i}}{\left(x+\beta_1 x+\gamma_1\right)^i} + \cdots + \sum_{i=1}^{q_t} \frac{B_{ti}x + C_{ti}}{\left(x+\beta_t x+\gamma_t\right)^i}$$

Where the as yet unknown coefficients $A_{ij}, B_{ij}, C_{ij}$ can be calculated from a system of linear algebraic equations obtained by first multiplying the equation by the denominator on the left and then comparing the coefficients of the polynomials on both sides of the equation.

<u>Example</u>

$$\frac{x^4-3x^3+2x-1}{x^7+17x^6+108x^5+322x^4+506x^3+540x^2+325x+125}=$$

$$=\frac{x^4-3x^3+2x-1}{\left(x+5\right)^3\left(x^2+x+1\right)^2}=$$

$$=\frac{A_{11}}{x+5}+\frac{A_{12}}{\left(x+5\right)^2}+\frac{A_{13}}{\left(x+5\right)^3}+\frac{B_{11}x+C_{11}}{x^2+x+1}+\frac{B_{12}x+C_{12}}{\left(x^2+x+1\right)^2}$$

$$x^4 - 3x^3 + 2x - 1 = A_{11}\left(x+5\right)^2\left(x^2+x+1\right)^2 + A_{11}\left(x+5\right)\left(x^2+x+1\right)^2 +$$

$$+A_{11}\left(x^2+x+1\right)^2 + \left(B_{11}x + C_{11}\right)\left(x+5\right)^3\left(x^2+x+1\right) +$$

$$+\left(B_{12}x + C_{12}\right)\left(x+5\right)^3$$

$$A_{11} = \frac{3830}{64827}, A_{12} = \frac{97}{343}, A_{13} = \frac{989}{441}$$

$$B_{11} = \frac{3830}{64827}, C_{11} = \frac{3013}{64827}, B_{12} = \frac{191}{3087}, C_{12} = \frac{-8}{3087}$$

All we need to do now to finish the task is to calculate the integrals of the following types:

1 $$\int \frac{A}{x-\alpha}\,dx$$

2 $$\int \frac{A}{(x-\alpha)^n}\,dx, n>1$$

3 $$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx$$

4 $$\int \frac{Bx+C}{\left(x^2+\beta x+\gamma\right)^n}\,dx, n>1$$

1

$$\int \frac{A}{x-\alpha}\,dx = \left| \begin{matrix} x-\alpha = t \\ dx = dt \end{matrix} \right| = A\int \frac{1}{t}\,dt = A\ln|t| = A\ln|x-\alpha| + c$$

2

$$\int \frac{A}{(x-\alpha)^n}\,dx = \left| \begin{matrix} x-\alpha = t \\ dx = dt \end{matrix} \right| = \int \frac{A}{t^n}\,dt = \frac{A}{(n-1)t^{n-1}} = \frac{A}{(n-1)(x-\alpha)^{n-1}} + c$$

3

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\int \frac{2x+\beta-\beta+\frac{2C}{B}}{x^2+\beta x+\gamma}\,dx =$$

$$= \frac{B}{2}\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx + \frac{B}{2}\int \frac{-\beta+\frac{2C}{B}}{x^2+\beta x+\gamma} =$$

$$= \frac{B}{2}\boxed{\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx} + \left(C-\frac{B\beta}{2}\right)\boxed{\int \frac{1}{x^2+\beta x+\gamma}\,dx}$$

3

$$\boxed{\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx} = \left| \begin{array}{c} x^2+\beta x+\gamma = t \\ (2x+\beta)\,dx = dt \end{array} \right| = \int \frac{1}{t}\,dt = \ln|t| =$$

$$= \ln\left|x^2+\beta x+\gamma\right|$$

**3**

$$\boxed{\int \frac{1}{x^2+\beta x+\gamma}\,dx} = \int \frac{1}{x^2+\beta x+\left(\beta/2\right)^2-\left(\beta/2\right)^2+\gamma}\,dx =$$

$$= \int \frac{1}{\left(x+\beta/2\right)^2+\left(4\gamma-\beta^2\right)/4}\,dx = \frac{4}{4\gamma-\beta^2}\int \frac{1}{\left(\dfrac{2x+\beta}{\sqrt{4\gamma-\beta^2}}\right)^2+1}\,dx =$$

$$= \left| \begin{array}{l} \dfrac{2x+\beta}{\sqrt{4\gamma-\beta^2}} = t \\ \dfrac{2dx}{\sqrt{4\gamma-\beta^2}} = dt \end{array} \right| = \frac{4}{4\gamma-\beta^2}\,\frac{\sqrt{4\gamma-\beta^2}}{2}\int \frac{1}{t^2+1}\,dx =$$

$$= \frac{2}{\sqrt{4\gamma-\beta^2}}\arctan t = \frac{2}{\sqrt{4\gamma-\beta^2}}\arctan\left(\frac{2x+\beta}{\sqrt{4\gamma-\beta^2}}\right)+c$$

3

The final result:

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{2C-B\beta}{\sqrt{4\gamma-\beta^2}}\arctan\frac{2x+\beta}{\sqrt{4\gamma-\beta^2}} + c$$

4 To show how to calculate $\int \frac{Bx+C}{\left(x^2+\beta x+\gamma\right)^n}dx$

we will use the following example:

$$\int \frac{x+1}{\left(x^2+1\right)}dx = \frac{1}{2}\int \frac{2x+2}{\left(x^2+1\right)}dx = \frac{1}{2}\int \frac{2x}{\left(x^2+1\right)}dx + \int \frac{1}{\left(x^2+1\right)}dx =$$

$$= \frac{-1}{2\left(x^2+1\right)} + \int \frac{1}{\left(x^2+1\right)^2}dx$$

Put $\int \frac{1}{\left(x^2+1\right)}dx = I_1$ and $\int \frac{1}{\left(x^2+1\right)^2}dx = I_2$

We know of course that $I_1 = \arctan x$ but suppose we try to integrate it by parts:

$$I_1 = \int \frac{1}{x^2+1}dx = \frac{x}{x^2+1} + 2\int \frac{x^2}{\left(x^2+1\right)^2}dx = \frac{x}{x^2+1} + 2\int \frac{x^2+1-1}{\left(x^2+1\right)^2}dx =$$

$$= \frac{x}{x^2+1} + 2I_1 - 2I_2$$

Thus $I_2 = \frac{1}{2}\left(\frac{x}{x^2+1} + I_1\right)$ and $I_2 = \frac{1}{2}\left(\frac{x}{x^2+1} + \arctan x\right)$