# Integrals of rational expressions in trigonometric functions

These are expressions of the type

$$R(\sin x, \cos x, \tan x, \cot x) = \frac{P(\sin x, \cos x, \tan x, \cot x)}{Q(\sin x, \cos x, \tan x, \cot x)}$$

where $P(x_1, x_2, x_3, x_4), Q(x_1, x_2, x_3, x_4)$ are polynomials.

Since $\tan x = \dfrac{\sin x}{\cos x}, \cot x = \dfrac{\cos x}{\sin x}$, we can write

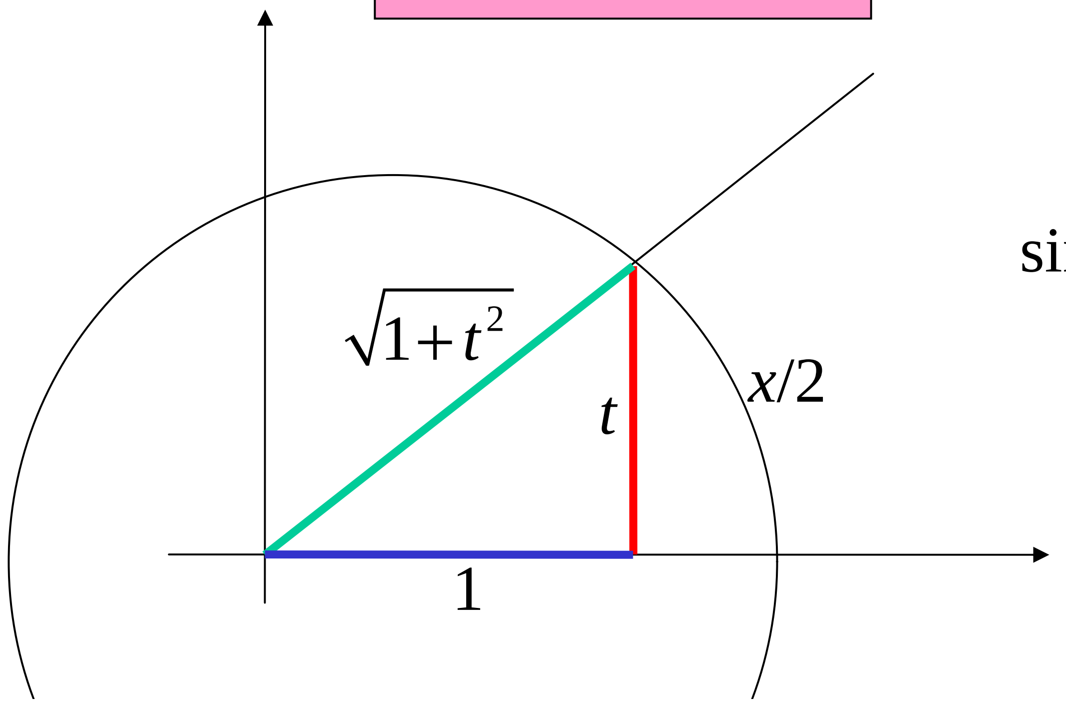
$$R(\sin x, \cos x, \tan x, \cot x) = \overline{R}(\sin x, \cos x) = \frac{\overline{P}(\sin x, \cos x)}{\overline{Q}(\sin x, \cos x)}$$

$$\int \overline{R}(\sin x, \cos x)\, dx$$

When calculating this integral, we can always use the universal substitution

$$\tan\frac{x}{2} = t$$

$$\sin\frac{x}{2} = \frac{t}{\sqrt{1-t^2}}, \quad \cos\frac{x}{2} = \frac{1}{\sqrt{1-t^2}}$$




$$\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

$$x = 2\arctan t$$

$$dx = \frac{2dt}{1+t^2}$$

Example

$$\int \frac{1}{1+\cos x}\,dx = \left| \begin{matrix} \tan\frac{x}{2} = t \\ dx = \frac{2dt}{1+t^2} \end{matrix} \right| = \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \frac{2dt}{1+t^2} = \int 1\,dt = t =$$

$$= \tan\frac{x}{2} + c = \sqrt{\frac{1-\cos x}{1+\cos x}} + c$$

If $\overline{R}(-\sin x, \cos x) = -\overline{R}(\sin x, \cos x)$ then the simpler substitution

$$\boxed{\cos x = t}$$

$$\sin x = \sqrt{1-t^2}$$

$$dx = \frac{-dt}{\sqrt{1-t^2}}$$

will do.

Similarly, if $\overline{R}(\sin x, -\cos x) = -\overline{R}(\sin x, \cos x)$, then

$$\boxed{\sin x = t}$$

$$\cos x = \sqrt{1-t^2}$$

$$dx = \frac{dt}{\sqrt{1-t^2}}$$

Example

$$\int \frac{1}{\cos x}\,dx = \left| \begin{array}{c} \sin x = t \\ dx = \dfrac{dt}{\sqrt{1-t^2}} \end{array} \right| = \int \frac{1}{\sqrt{1-t^2}}\,\frac{dt}{\sqrt{1-t^2}} = \int \frac{dt}{1-t^2} =$$

$$= \frac{1}{2}\int \frac{1}{1-t} + \frac{1}{1+t}\,dt = \frac{1}{2}\left(-\ln|t-1| + \ln|t+1|\right) = \ln\sqrt{\frac{t+1}{t-1}} =$$

$$= \ln\sqrt{\frac{\sin x + 1}{\sin x - 1}} + c$$

Integrals of the type

$$\int R\left(\left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}},\left(\frac{ax+b}{cx+d}\right)^{\frac{p_2}{q_2}},\ldots,\left(\frac{ax+b}{cx+d}\right)^{\frac{p_s}{q_s}},x\right)dx$$

where

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,\ p_i, q_i$ are relatively prime positive integers for $i = 1, 2, \ldots, s$

can be calculated using the substitution

$\left(\frac{ax+b}{cx+d}\right)^{\frac{1}{q}} = t$ where $q$ is the least common multiple of $q_1, q_2, \ldots, q_s$

Example

$$\int \sqrt{\frac{x+1}{x-1}} + \sqrt[3]{\frac{x+1}{x-1}}\, dx = \left| \begin{array}{c} \sqrt[6]{\dfrac{x+1}{x-1}} = t \\ x = \dfrac{t^6+1}{t^6-1} \\ dx = \dfrac{-12t^5}{\left(t^6-1\right)^2} \end{array} \right| = \int \left(t^3+t^2\right) \frac{-12t^5}{\left(t^6-1\right)^2}\, dt$$

# Euler substitutions

When calculating integrals of the type

$$\int R\left(\sqrt{ax^2+bx+c}\,,x\right)dx$$

where $R(y,x)$ is a rational expression, we can use Euler substitutions that transform the integral into

$$\int \overline{R}(t)\,dt$$

where $\overline{R}(t)$ is a rational expression.

There are three types of Euler substitutions.

# Type 1

$$\int R\left(\sqrt{ax^2+bx+c},x\right)dx$$

$$\boxed{a>0}$$

$$\sqrt{ax^2+bx+c}=t-\sqrt{a}x$$

$$ax^2+bx+c=t^2-2\sqrt{a}xt+ax^2$$

$$x=\frac{t^2-c}{b+2\sqrt{a}t} \qquad dx=2\frac{bt+\sqrt{a}c}{\left(b+2\sqrt{a}t\right)^2}$$

Type 2

$$\int R\left(\sqrt{ax^2+bx+c},x\right)dx$$

$$c>0$$

$$\sqrt{ax^2+bx+c}=xt+\sqrt{c}$$

$$ax^2+bx+c=x^2t^2+2\sqrt{c}xt+c$$

$$x=\frac{2\sqrt{c}t-b}{a-t^2}$$

$$dx=\frac{1}{2}\frac{\sqrt{c}t^2-bt+a\sqrt{c}}{\left(a-t^2\right)^2}$$

Type 3

$$\int R\left(\sqrt{ax^2+bx+c},x\right)dx \qquad \boxed{b^2-4ac>0}$$

$$ax^2+bx+c=a(x-u)(x-v)$$

$$\sqrt{ax^2+bx+c}=t(x-u)$$

$$a(x-u)(x-v)=t^2(x-u)^2 \Rightarrow a(x-v)=t^2(x-u)$$

$$x=\frac{t^2u-av}{t^2-a} \qquad dx=\frac{2ta(v-u)}{\left(t^2-a\right)^2}dt$$

Another method of calculating integrals of the type

$$\int R\left(\sqrt{ax^2+bx+c}, x\right) dx$$

By completing the square under the square root sign, we eventually arrive at the following three inegral types:

- 1) $\int R\left(\sqrt{m^2-z^2}, z\right) dz$
- 2) $\int R\left(\sqrt{m^2+z^2}, z\right) dz$
- 3) $\int R\left(\sqrt{z^2-m^2}, z\right) dz$

- 1) $\int R\left(\sqrt{m^2 - z^2}, z\right) dz$

Substituting

$$z = m\sin t,\ dz = m\cos t$$

yields

$$m\int R\left(m\cos t, \sin t\right)\cos t\ dt$$

2) $\int R\left(\sqrt{m^2+z^2}, z\right) dz$

Substituting

$$z = m\tan t,\ dz = \frac{m}{\cos^2 t}$$

yields

$$m\int \frac{R\left(\dfrac{m}{\cos t}, \sin t\right)}{\cos^2 t}\, dt$$

3) $\int R\left(\sqrt{z^2 - m^2}, z\right) dz$

Substituting

$$z = \frac{m}{\cos t}, \; dz = \frac{m \sin t}{\cos^2 t}$$

yields

$$m\int R\left(m\tan t, \frac{m}{\cos t}\right)\frac{\sin t}{\cos^2 t}\, dt$$

## Problems to solve

1. $$\int \sqrt{3-2x-x^2}\, dx$$

2. $$\int \sqrt{2+x^2}\, dx$$

3. $$\int \frac{x^2}{\sqrt{9+x^2}}\, dx$$

4. $$\int \sqrt{x^2-2x+2}\, dx$$

5. $$\int \sqrt{x^2-4}\, dx$$

6. $$\int \sqrt{x^2+x}\, dx$$

7. $$\int \sqrt{x^2-6x-7}\, dx$$

8. $$\int \frac{1}{(x-1)\sqrt{x^2-3x+2}}\, dx$$

9. $$\int \frac{1}{(1+x^2)\sqrt{1-x^2}}\, dx$$

10. $$\int \frac{1}{(1-x^2)\sqrt{1+x^2}}\, dx$$

## Results

1. $$\frac{x+1}{2}\sqrt{3-2x-x^2}+2\arcsin\frac{x+1}{2}$$

2. $$\frac{x}{2}\sqrt{2+x^2}+\ln\left(x+\sqrt{2+x^2}\right)$$

3. $$\frac{x}{2}\sqrt{9+x^2}-\frac{9}{2}\ln\left(x+\sqrt{9+x^2}\right)$$

4. $$\frac{x-1}{2}\sqrt{x^2-2x+2}+\frac{1}{2}\ln\left(x-1+\sqrt{x^2-2x+2}\right)$$

5. $$\frac{x}{2}\sqrt{x^2-4}-2\ln\left|x+\sqrt{x^2-4}\right|$$

## Results

6 $$\frac{2x+1}{4}\sqrt{x^2+x}-\frac{1}{8}\ln\left|2x+1+2\sqrt{x^2+x}\right|$$

7 $$\frac{x-3}{2}\sqrt{x^2-6x-7}-8\ln\left|x-3+\sqrt{x^2-6x-7}\right|$$

8 $$2\sqrt{\frac{x-2}{x-1}}$$

9 $$\frac{1}{\sqrt{2}}\arctan\frac{x\sqrt{2}}{\sqrt{1-x^2}}$$

10 $$\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{1+x^2}+x\sqrt{2}}{\sqrt{1+x^2}-x\sqrt{2}}\right|$$