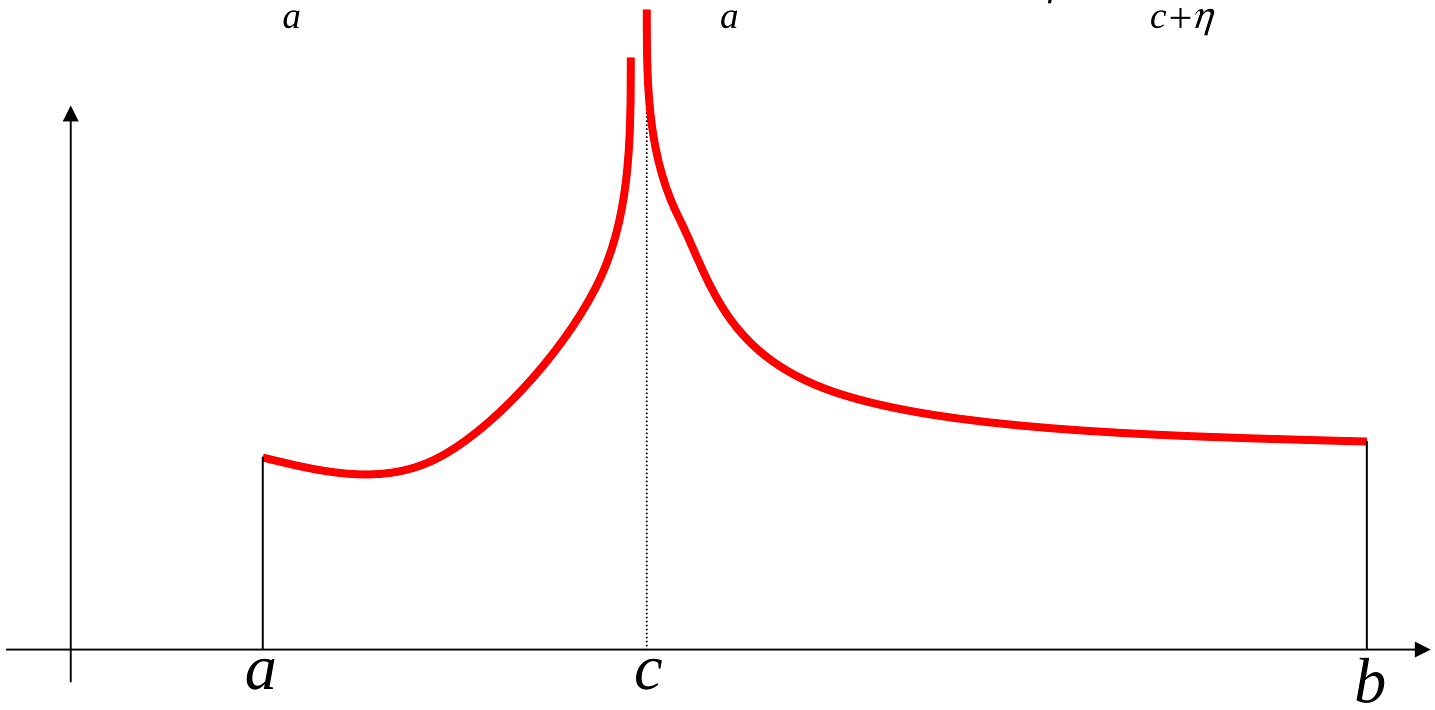
Integral of an unbounded function

If $f(x)$ is unbounded in any neighbourhood of a point $c \in [a,b]$ and continuous for $a \le x < c,\ c < x \le b$ then we put

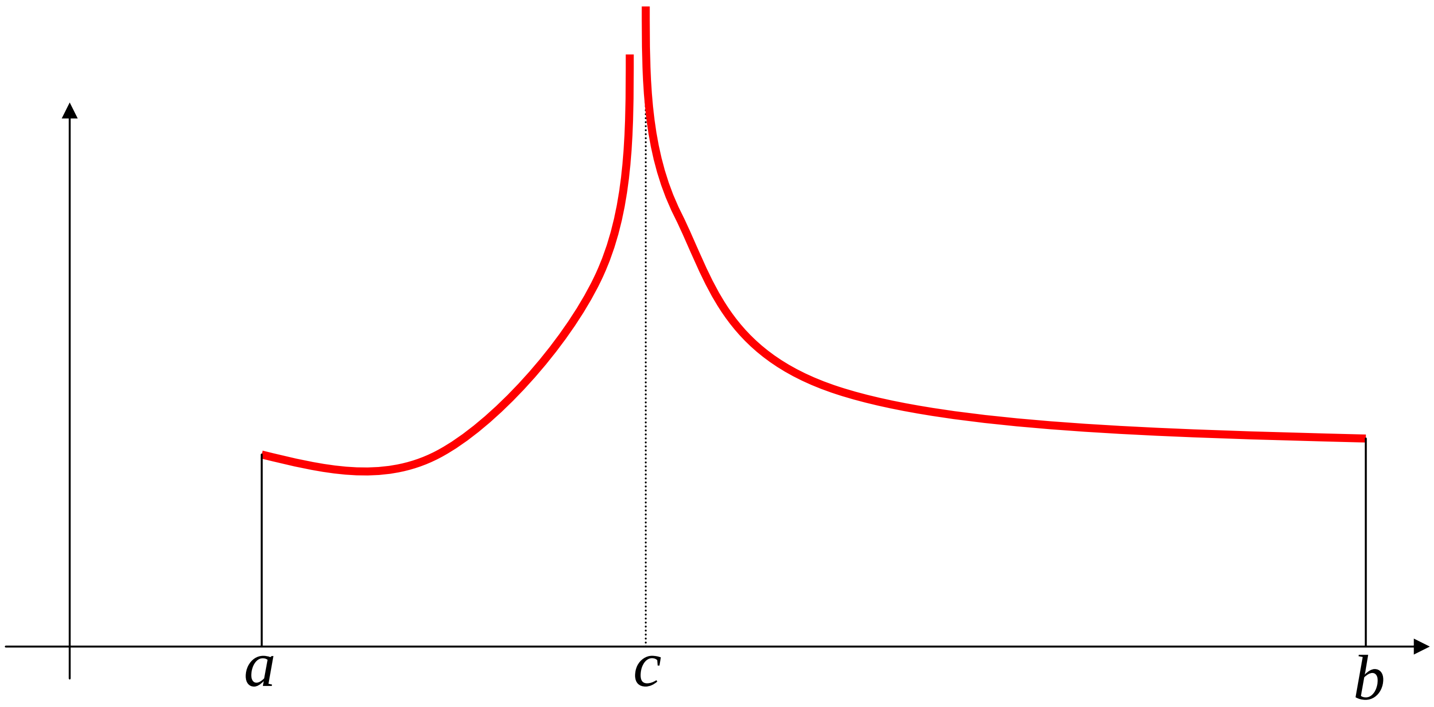
$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\eta \to 0+} \int_{c+\eta}^{b} f(x)\,dx$$



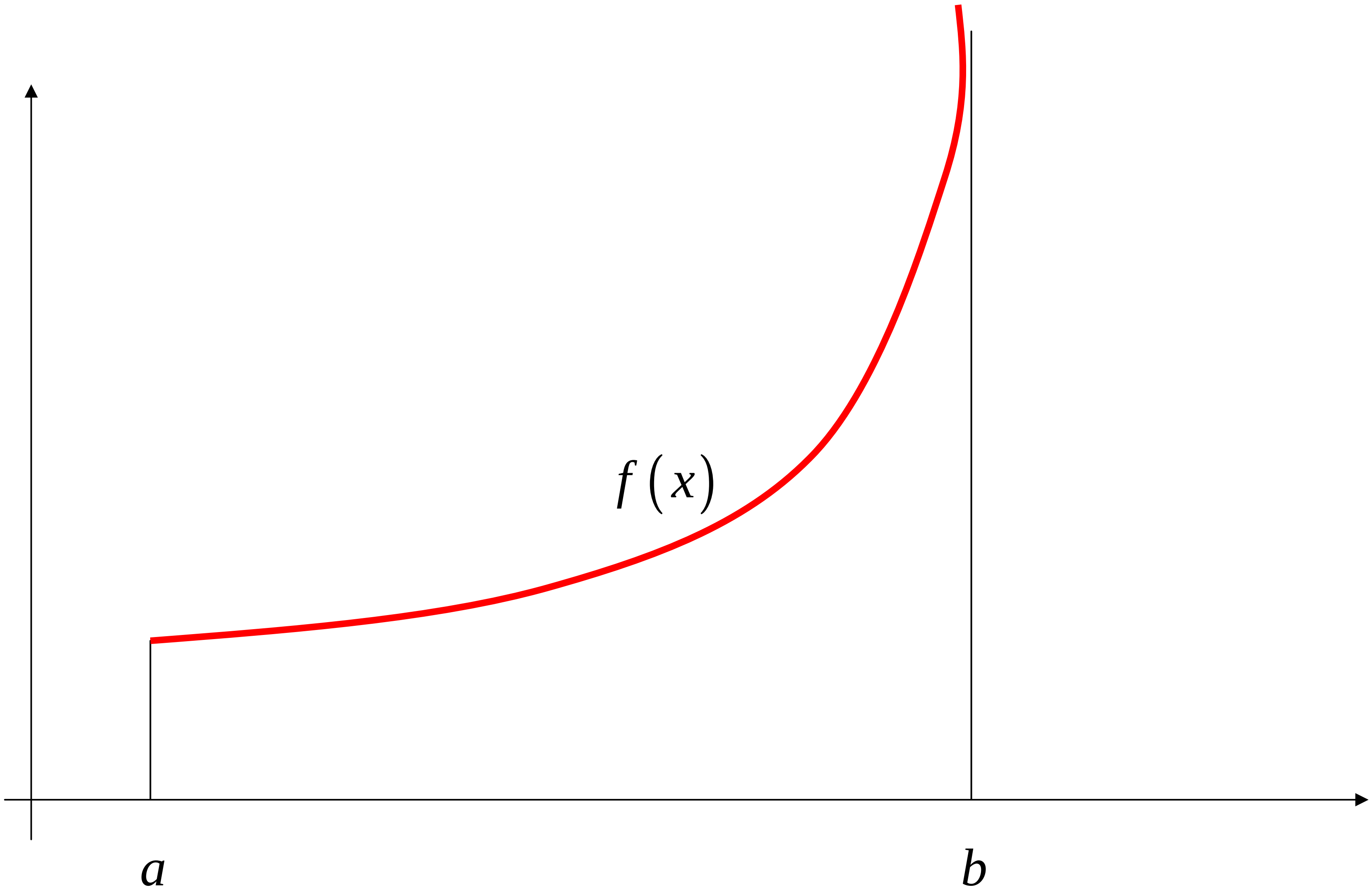
If the limits $\lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)\,dx$ and $\lim_{\eta \to 0+} \int_{c+\eta}^{b} f(x)\,dx$

exist, we say that the integral $\int_a^b f(x)\,dx$ is convergent or converges

otherwise we say that it diverges or is divergent.



If $c = a$ or $c = b$, the integral is defined in much the same way



If a function $F(x)$ exists continuous on $[a,b]$ such that

$F'(x) = f(x)$ for $x \in [a,b], x \neq c$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

If $|f(x)| \leq \Phi(x), a \leq x \leq b$ then if $\int_a^b \Phi(x)\,dx$ converges

so does $\int_a^b f(x)\,dx$

If $f(x) \geq 0$ and $\lim_{x \to c} \left\{ f(x) |c - x|^m \right\} = A \neq \infty, A \neq 0$ that is

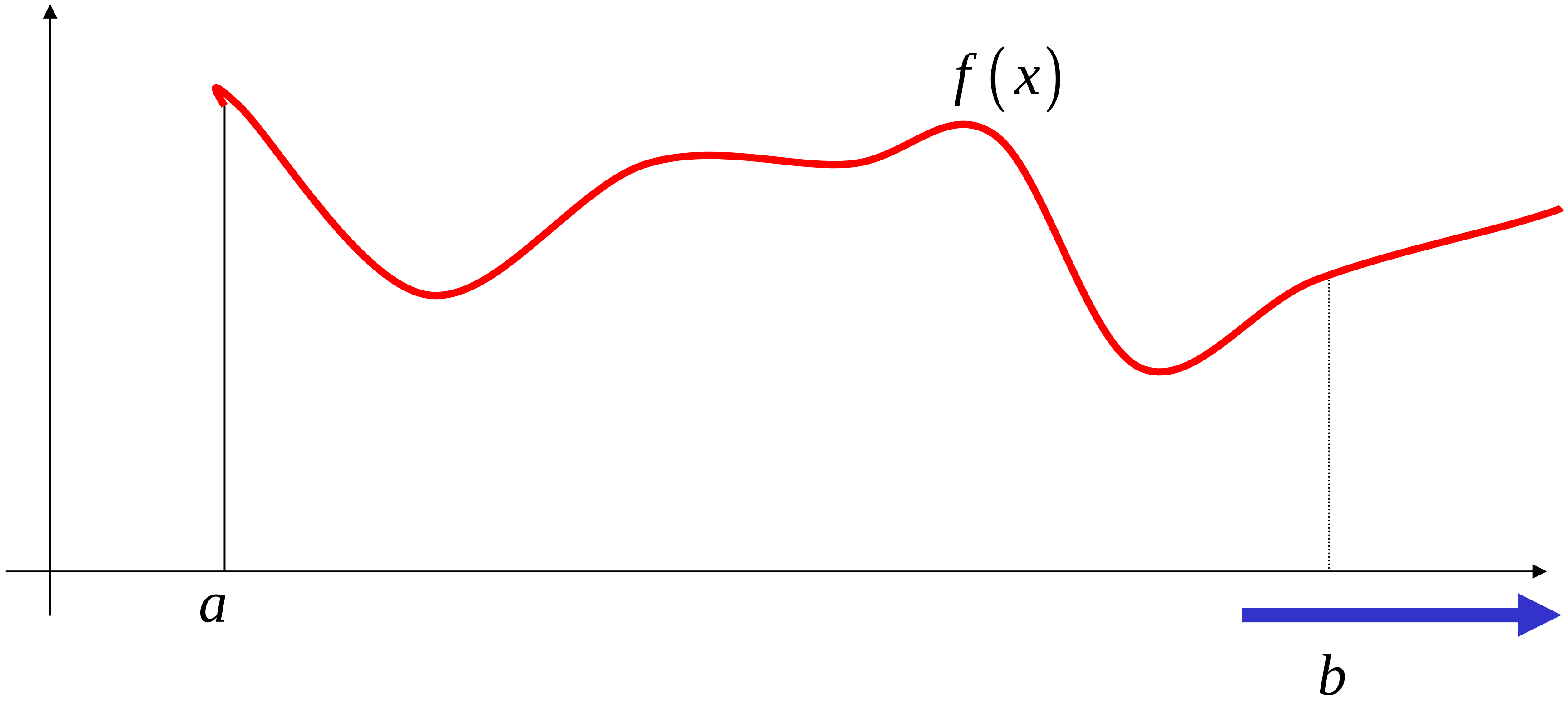
$f(x) \approx \dfrac{A}{|c-x|^m}$ as $x \to c$ then

- $\int_a^b f(x)\, dx$ converges for $m < 1$
- $\int_a^b f(x)\, dx$ diverges for $m \geq 1$

Integrals over unbounded intervals

If a function $f(x)$ is continuous for $a \le x < \infty$ then we put

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^{b} f(x)\,dx$$



If the limit $\lim\limits_{b\to\infty}\int_a^b f(x)\,dx$ exists, then we say that $\int_a^\infty f(x)\,dx$

converges or is convergent, otherwise we say that it diverges or is divergent.

By analogy, we put $\int_{-\infty}^b f(x)\,dx = \lim\limits_{a\to-\infty}\int_a^b f(x)\,dx$

and $\int_{-\infty}^\infty f(x)\,dx = \lim\limits_{b\to-\infty}\left(\lim\limits_{a\to-\infty}\int_a^b f(x)\,dx\right)$

If $\left|f(x)\right| \leq \Phi(x), a \leq x$ then if $\int_a^\infty \Phi(x)\,dx$ converges

so does $\int_a^\infty f(x)\,dx$

If $f(x) \geq 0$ and $\lim_{x \to c} \left\{ f(x) x^m \right\} = A \neq \infty, A \neq 0$ that is

$f(x) \approx \dfrac{A}{x^m}$ as $x \to \infty$ then

- $\int_a^\infty f(x)\,dx$ converges for $m > 1$
- $\int_a^\infty f(x)\,dx$ diverges for $m \leq 1$

Example

$$\int_{-1}^{1} \frac{dx}{x^2}\, dx = \lim_{\varepsilon\to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2}\, dx + \lim_{\eta\to 0} \int_{\eta}^{1} \frac{dx}{x^2}\, dx = \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}-1\right) + \lim_{\eta\to 0}\left(\frac{1}{\eta}-1\right) = \infty$$

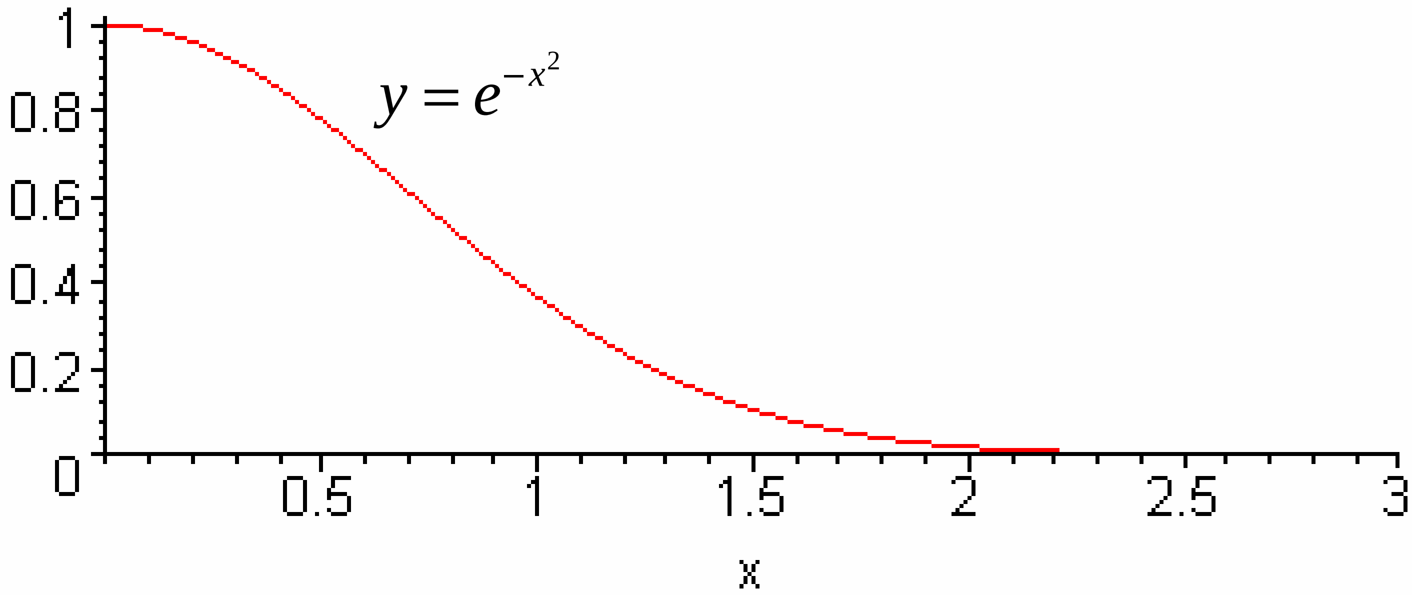
The integral diverges.

Example

$$\int_0^{\infty} \frac{dx}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^{b} \frac{dx}{1+x^2}\,dx = \lim_{b\to\infty}\left(\arctan b - \arctan 0\right) = \frac{\pi}{2}$$

Example

Determine whether the Euler-Poisson integral $\int_0^\infty e^{-x^2}\,dx$ converges.



Put $$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

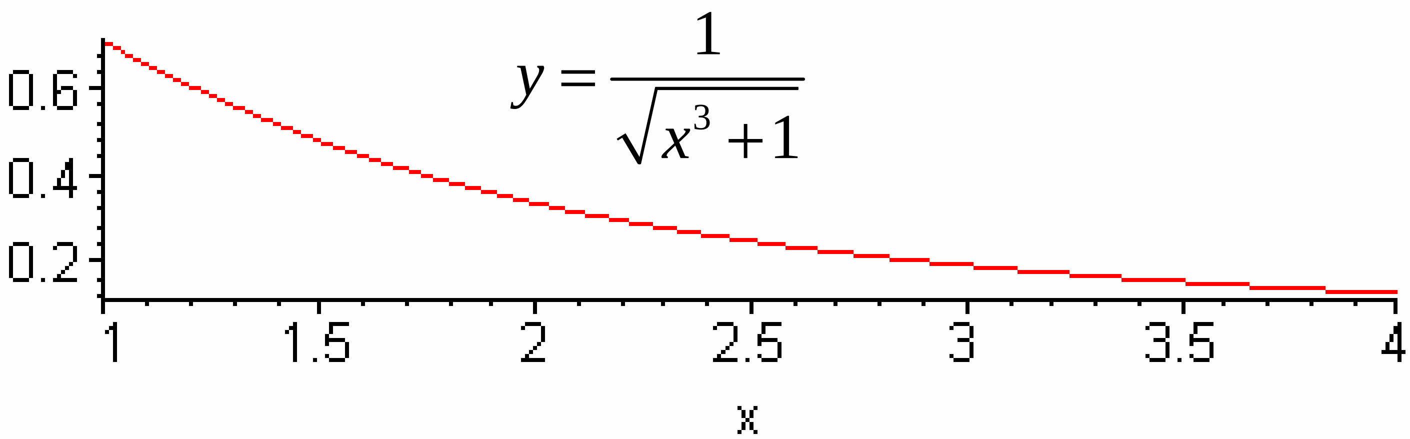
The first integral on the right-hand side is not improper and the second one is convergent since we have

$$e^{-x^2} \le e^{-x} \text{ for } x \ge 1 \quad \text{and}$$

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}\left(-e^{-b}+e^{-1}\right) = e^{-1}$$

Example

Determine whether the integral $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}\,dx$ converges or not.



As $x \to \infty$ we have

$$\frac{1}{\sqrt{x^3+1}} = \frac{1}{\sqrt{x^3\left(1+\frac{1}{x^3}\right)}} = \frac{1}{x^{3/2}}\frac{1}{\sqrt{1+\frac{1}{x^3}}} \to \frac{1}{x^{3/2}}$$

Since the integral $\int_1^\infty \frac{dx}{x^{\frac{3}{2}}}\,dx$ converges, so does $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}\,dx$

Example

Determine whether the integral $\int_0^1 \frac{dx}{\sqrt{1-x^4}}\,dx$ converges or not.

At $x = 1$, the integral is discontinuous. Using the formula

$$1-x^4 = (1-x)(1+x)\left(1+x^2\right)$$

we obtain

$$\frac{1}{\sqrt{1-x^4}} = \frac{1}{\sqrt{(1-x)(1+x)\left(1+x^2\right)}} = \frac{1}{(1-x)^{\frac{1}{2}}}\frac{1}{\sqrt{(1+x)\left(1+x^2\right)}}$$

This means that, as $x \to 1$, we get

$$\frac{1}{\sqrt{1-x^4}} \to \frac{1}{2}\left(\frac{1}{1-x}\right)^{1/2}$$

Since $\displaystyle\int_0^1 \left(\frac{1}{1-x}\right)^{\frac{1}{2}} dx$ converges, so does $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}}\, dx$