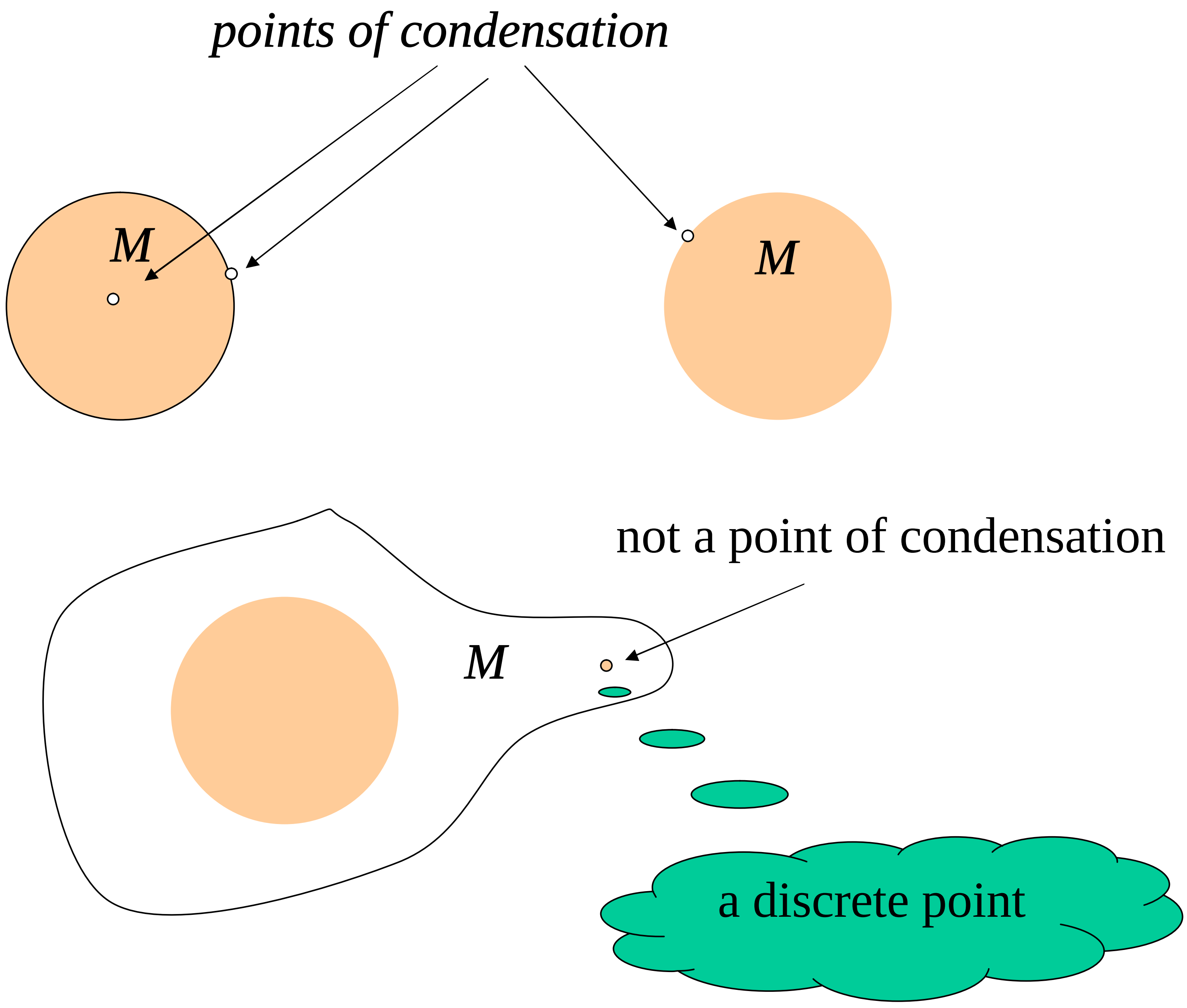
Points of condensation

Let $M \subseteq E_n$ and $X = \left[x_1, x_2, \mathbf{K}, x_n\right] \in M$

We say that X is a point of condensation of M if every

neighbourhood of X contains at least one point of M.

$$\forall e > 0 : \exists Y \in M : Y \in N\left(X, e\right)$$


points of condensation
M
M
not a point of condensation
M
a discrete point

Let $f\left(x_1, x_2, \mathbf{K}, x_n\right)$ be defined on M and let $A = \left[a_1, a_2, \mathbf{K}, a_n\right] \in M$

be a point if condensation. We say that $b \in R$ is the **limit** of

$f\left(x_1, x_2, \mathbf{K}, x_n\right)$ at A if, for every $e > 0$

there exists a neighbourhood $N\left(A, d\right), d > 0$ such that

for all $X = \left[x_1, x_2, \mathbf{K}, x_n\right] \in N\left(A, d\right)$ we have

$$\left| f\left(x_1, x_2, \mathbf{K}, x_n\right) - b \right| < e$$

We use the following denotations:

$$\lim_{X \to A} f(X) = b$$

$$\lim_{\substack{x_1 \to a_1 \\ x_2 \to a_2 \\ \vdots \\ x_n \to a_n}} f(X) = b$$

Some basic properties of limits of functions of one variable also extend to limits of functions of more variables such as

- $$\lim_{X \to A}\left(f(x) \pm g(x) \right) = \lim_{X \to A} f(x) \pm \lim_{X \to A} g(x)$$
- $$\lim_{X \to A}\left(f(x) \cdot g(x) \right) = \lim_{X \to A} f(x) \cdot \lim_{X \to A} g(x)$$
- $$\lim_{X \to A}\left(f(x) / g(x) \right) = \lim_{X \to A} f(x) / \lim_{X \to A} g(x) \quad \lim_{X \to A} g(x) \neq 0$$
- $\lim_{X \to A} c = c,$ where c is a constant
- $$\lim_{X \to A}\left(k \cdot f(x) \right) = k \lim_{X \to A} f(x)$$

Provided that the right-hand-side limits exist

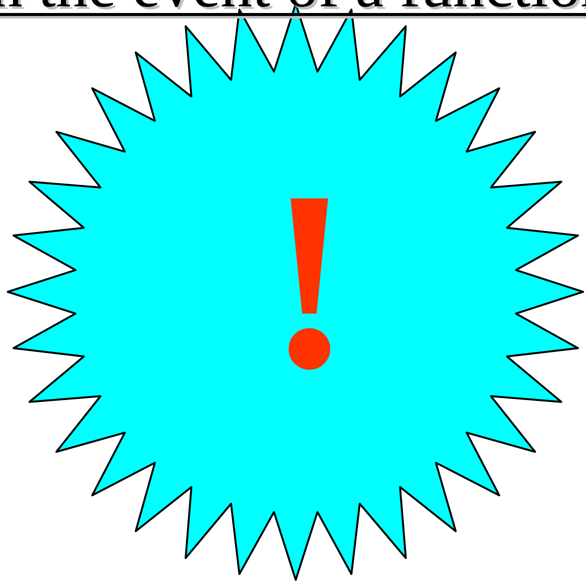
Also a parallel of the squeezing lemma can be formulated:

If $\lim_{X \to A} f(X) = b$ and $\lim_{X \to A} g(X) = b$ and there exists a neighbourhood $N(A,d)$ of A such that, for $X \in N(A,d)$, we have, for a function $h(X)$, $f(X) < h(X) < g(X)$ then

$$\lim_{X \to A} h(X) = b$$

Similarly, it can be proved that, if a function has a limit at a point *A*, then the limit is unique.

However, finding the limit of a function of *n* variables or even proving that a function of *n* variables has a limit is a much more difficult task than in the event of a function of one variable.



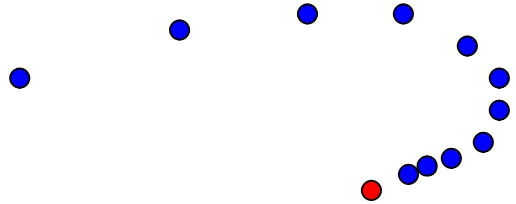
We will now develop some means that can help us in this difficult task and apply them to functions of two variables.

Let $X_1, X_2, \mathbf{K}, X_n, \mathbf{K}$ be an infinite sequence of points in E_n.

We say that $X_1, X_2, \mathbf{K}, X_n, \mathbf{K}$ converges to a point A if,

for every $e > 0$, there is an index N such that, for $n > N$,

we have $r\left(X_n, A\right) < e$ Formally, we write



$$X_1, X_2, \mathbf{K}, X_n, \mathbf{K} \to A$$

or

$$\lim_{n \to \infty} X_i = A$$

The following two assertions are equivalent:

- $\lim\limits_{X \to A} f\left(X\right) = b$

- For every sequence $X_1, X_2, \mathbf{K}, X_n, \mathbf{K}$ of points in $D(f)$ such that $X_1, X_2, \mathbf{K}, X_n, \mathbf{K} \to A$ we have $\lim\limits_{n \to \infty} f\left(X_n\right) = b$

Example

Prove that the function $f(x,y)=\dfrac{x^2y}{x^2+y^2}$ has a limit at [0,0]

$$0 \le \left| \frac{x^2y}{x^2+y^2} \right| \le |y|$$

$$-y \le \frac{x^2y}{x^2+y^2} \le y$$

$$\lim_{y\to 0} y = \lim_{y\to 0}(-y) = 0$$

Example

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x-y}{x+y}$$ This limit does not exist:

$$\left[\frac{1}{2},\frac{1}{2}\right],\left[\frac{1}{4},\frac{1}{4}\right],\mathbf{K},\left[\frac{1}{2^n},\frac{1}{2^n}\right]\to[0,0] \qquad \frac{\frac{1}{2^n}-\frac{1}{2^n}}{\frac{1}{2^n}+\frac{1}{2^n}}=0\cdot 2^{n-1}\to 0$$

$$\left[0,\frac{1}{2}\right],\left[0,\frac{1}{4}\right],\mathbf{K},\left[0,\frac{1}{2^n}\right]\to[0,0] \qquad \frac{0-\frac{1}{2^n}}{0+\frac{1}{2^n}}=-1\to -1$$

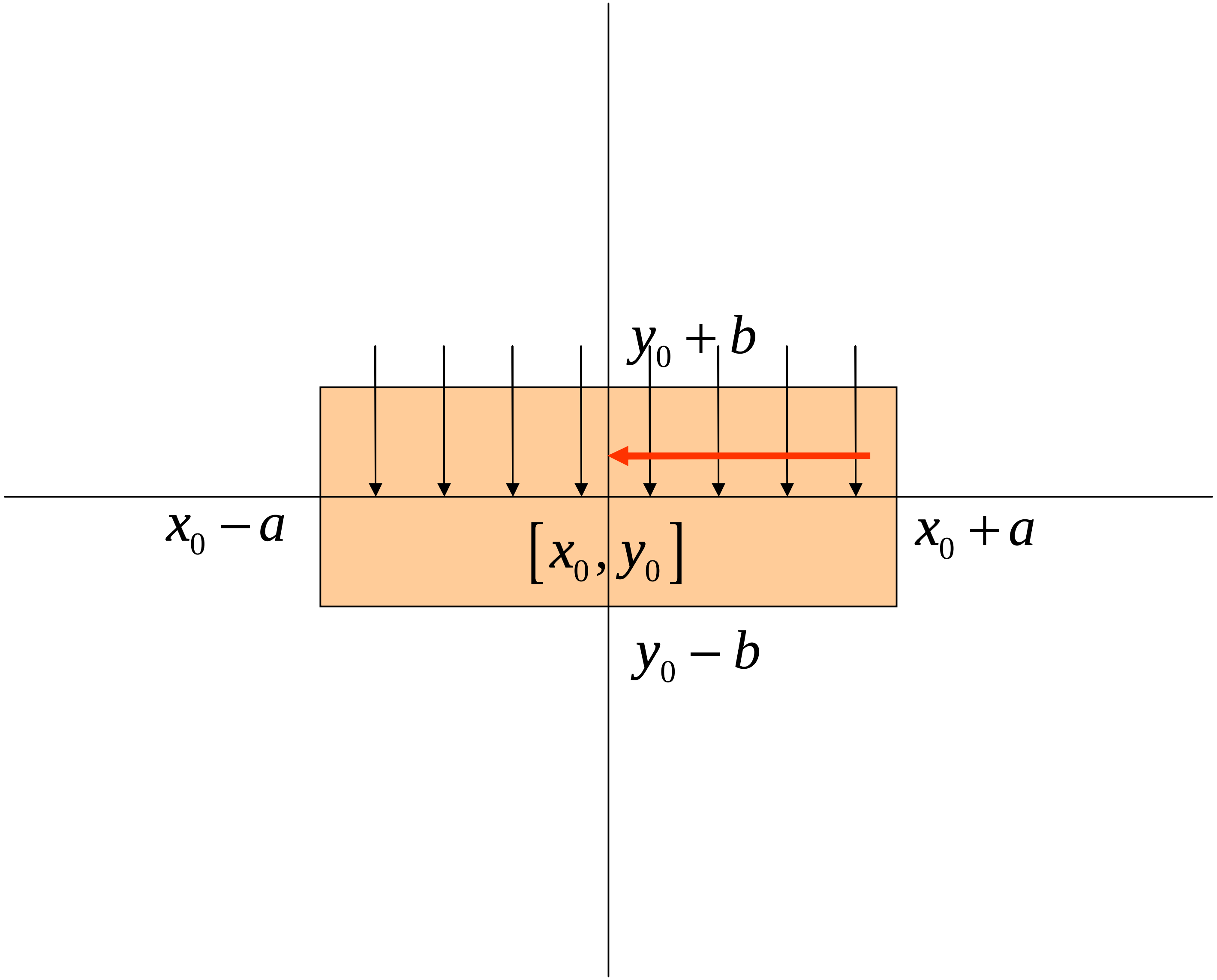
Let $z = f(x, y)$ be a function of two variables defined on the set

$[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ and let the limit $\lim\limits_{y \to y_0} f(x, y)$

be defined for every $x \in [x_0 - a, x_0 + a]$. Thus we can define

a function $j(x) = \lim\limits_{y \to y_0} f(x, y)$ If $\lim\limits_{x \to x_0} j(x) = A$ then the

number A is called the double limit of $f(x, y)$ at $[x_0, y_0]$


$y_0 + b$
$x_0 - a$
$[x_0, y_0]$
$x_0 + a$
$y_0 - b$

If

1. $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) = A$ where $f(x,y)$ is defined on a rectangle

   $$[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$$

2. for any $y \in [y_0 - b, y_0 + b]$ there exists a limit

   $$j(y) = \lim_{x\to x_0} f(x,y)$$

Then the double limit $\lim\limits_{y\to y_0} j(y)$ exists and $\lim\limits_{y\to y_0} j(y) = A$

Example

$$\lim_{\substack{x\to 0\\ y\to 0}} x\sin\frac{1}{y}$$

$$\left|x\sin\frac{1}{y}\right| \le |x| \;\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} x\sin\frac{1}{y} = 0$$

for $y = 0$, no limit for $x \to 0$ exists

Counterexample showing that if condition 1 is satisfied in the above theorem condition 2 need not be.

<u>Example</u>

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$$

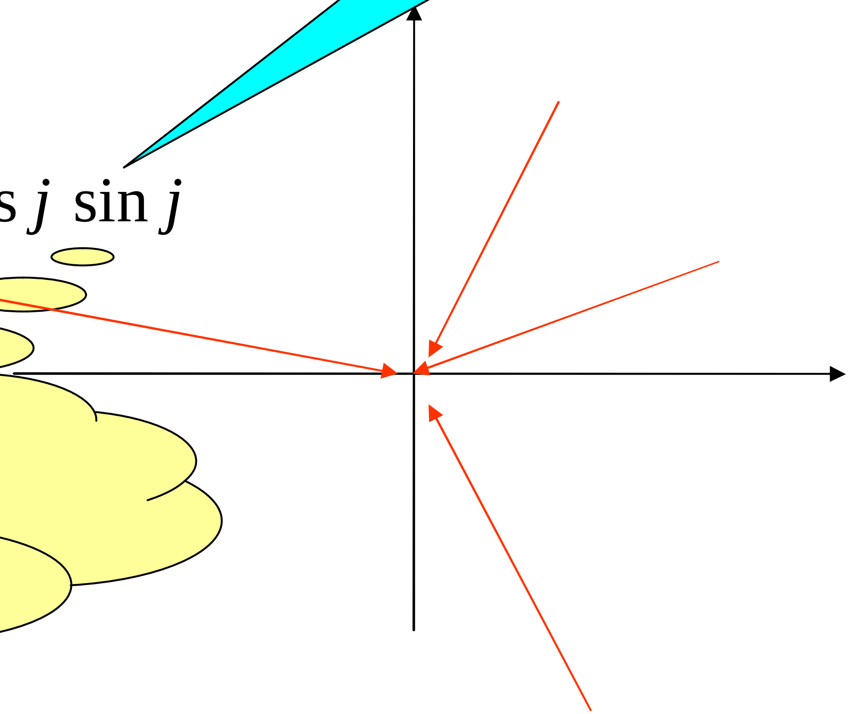
Reverse assertion may be false

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy}{x^2+y^2}\right)=\lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy}{x^2+y^2}\right)=0$$

No limit exists

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}=\lim_{r\to 0}\frac{r^2\cos\varphi\,\sin\varphi}{r^2}=\cos\varphi\,\sin\varphi$$

different results for different approaching angles



We say that a function $f(X) = f(x_1, x_2, \mathbf{K}, x_n)$ is continuous at A, $A = [a_1, a_2, \mathbf{K}, a_n]$ if it is defined at A and has a limit that is equal to $f(a_1, a_2, \mathbf{K}, a_n)$

If $f\left(x_1, x_2, \mathbf{K}, x_n\right)$ and $g\left(x_1, x_2, \mathbf{K}, x_n\right)$ are continuous at

$\left[a_1, a_2, \mathbf{K}, a_n\right]$ then so are

$$f\left(x_1, x_2, \mathbf{K}, x_n\right) + g\left(x_1, x_2, \mathbf{K}, x_n\right)$$

$$f\left(x_1, x_2, \mathbf{K}, x_n\right) - g\left(x_1, x_2, \mathbf{K}, x_n\right)$$

$$f\left(x_1, x_2, \mathbf{K}, x_n\right) \cdot g\left(x_1, x_2, \mathbf{K}, x_n\right)$$

$$f\left(x_1, x_2, \mathbf{K}, x_n\right) / g\left(x_1, x_2, \mathbf{K}, x_n\right)$$

if $g\left(a_1, a_2, \mathbf{K}, a_n\right) \neq 0$

If $f\left(y_1, y_2, \mathbf{K}, y_m\right)$ and

$$u_1\left(x_1, x_2, \mathbf{K}, x_n\right), u_2\left(x_1, x_2, \mathbf{K}, x_n\right), \mathbf{K}, u_m\left(x_1, x_2, \mathbf{K}, x_n\right)$$

are continuous at $\left[a_1, a_2, \mathbf{K}, a_n\right]$, then so is the composite function

$$h\left(x_1, x_2, \mathbf{K}, x_n\right) = f\left(u_1\left(x_1, x_2, \mathbf{K}, x_n\right), u_2\left(x_1, x_2, \mathbf{K}, x_n\right), \mathbf{K}, u_m\left(x_1, x_2, \mathbf{K}, x_n\right)\right)$$

Note that , if a function $f\left(x_1, x_2, \ldots, x_n\right)$ is continuous at $\left[a_1, a_2, \ldots, a_n\right]$ its limit at $\left[a_1, a_2, \ldots, a_n\right]$ is calculated simply by substituting $\left[a_1, a_2, \ldots, a_n\right]$ into $f\left(x_1, x_2, \ldots, x_n\right)$

$$\lim_{\substack{x \to x_1 \\ x \to x_2 \\ \vdots \\ x \to x_n}} f\left(x_1, x_2, \ldots, x_n\right) = f\left(a_1, a_2, \ldots, a_n\right)$$