Let a function $f\left(x_1, x_2, \mathbf{K}, x_n\right)$ be differentiable at a point

$A=\left[a_1, a_2, \mathbf{K}, a_n\right] \in D\left(f\right)$ and let functions $j_1\left(t\right), j_2\left(t\right), \mathbf{K}, j_n\left(t\right)$

be defined on [*a*,*b*] such that they have a derivative at a $t_0 \in \left[a, b\right]$

Let further $j_1\left(t_0\right)=a_1, j_2\left(t_0\right)=a_2, \mathbf{K}, j_n\left(t_0\right)=a_n.$

Then the function $h\left(t\right)= f\left(j_1\left(t\right), j_2\left(t\right), \mathbf{K}, j_n\left(t\right)\right)$

has a derivative at $t=t_0$ with

$$h'\left(t_0\right)= f'_{x_1}\left(j_1\left(t_0\right), j_2\left(t_0\right), \mathbf{K}, j_n\left(t_0\right)\right) j'_1\left(t_0\right)+\mathbf{L}$$
$$\mathbf{L}+ f'_{x_n}\left(j_1\left(t_0\right), j_2\left(t_0\right), \mathbf{K}, j_n\left(t_0\right)\right) j'_n\left(t_0\right)$$

Let $f\left(y_1, y_2, \mathbf{K}, y_m\right)$ be differentiable at $Y = \left[b_1, b_2, \mathbf{K}, b_m\right] \in D\left(f\right)$

and let the functions $j_i\left(x_1, x_2, \mathbf{K}, x_n\right), i = 1, 2, \mathbf{K}, m$ all have

a partial derivative $\dfrac{\partial j_i\left(x_1, x_2, \mathbf{K}, x_n\right)}{\partial x_j}$ at a point

$$X = \left[a_1, a_2, \mathbf{K}, a_n\right] \in D\left(j_1\right) \cap D\left(j_2\right) \cap \mathbf{K} \cap D\left(j_m\right)$$

Let further $j_i\left(a_1, a_2, \mathbf{K}, a_n\right) = b_i, i = 1, 2, \mathbf{K}, m$ and define

$$h\left(x_1, x_2, \mathbf{K}, x_n\right) = f\left(j_1\left(x_1, x_2, \mathbf{K}, x_n\right), \mathbf{K}, j_m\left(x_1, x_2, \mathbf{K}, x_n\right)\right)$$

Then $h\left(x_1, x_2, \mathbf{K}, x_n\right)$ has a partial derivative by x_j at X with

$$\frac{\partial h\left(X\right)}{\partial x_j} = \frac{\partial f\left(Y\right)}{\partial y_1}\frac{\partial j_1\left(X\right)}{\partial x_j} + \frac{\partial f\left(Y\right)}{\partial y_2}\frac{\partial j_2\left(X\right)}{\partial x_j} + \mathbf{L} + \frac{\partial f\left(Y\right)}{\partial y_m}\frac{\partial j_m\left(X\right)}{\partial x_j}$$

Example

Find $\dfrac{du(t)}{dt}$ if $u = \ln\cos\dfrac{x}{\sqrt{y}}$ and $x = 3t, y = \sqrt{t^2+9}$ at $t = 0$

$$\frac{du(t)}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\cos\frac{x}{\sqrt{y}}}\left(-\sin\frac{x}{\sqrt{y}}\right)\frac{1}{\sqrt{y}} = \frac{-\tan\frac{x}{\sqrt{y}}}{\sqrt{y}}$$

$$x_t^{'} = 3, y_t^{'} = \frac{t}{\sqrt{t^2+9}}$$

$$\frac{\partial u}{\partial x} = -\tan\frac{x}{\sqrt{y}}\frac{-1}{2}\frac{x}{\sqrt{y^3}} = \frac{x\tan\frac{x}{\sqrt{y}}}{2\sqrt{y^3}}$$

$$\boxed{\frac{du(0)}{dt} = 0}$$

Example

$$z = \arctan\frac{x}{y},\ x = u\sin v,\ y = u\cos v \text{ at } u = 1, v = 0$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad x = 0, y = 1$$

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y}, \frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{-x}{y^2} \Rightarrow \frac{\partial z}{\partial x} = 1, \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial x}{\partial u} = \sin v, \frac{\partial y}{\partial u} = \cos v \Rightarrow \frac{\partial x}{\partial u} = 0, \frac{\partial y}{\partial u} = 1$$

$$\boxed{\frac{\partial z}{\partial u} = 1\cdot 0 + 0\cdot 1 = 0}$$