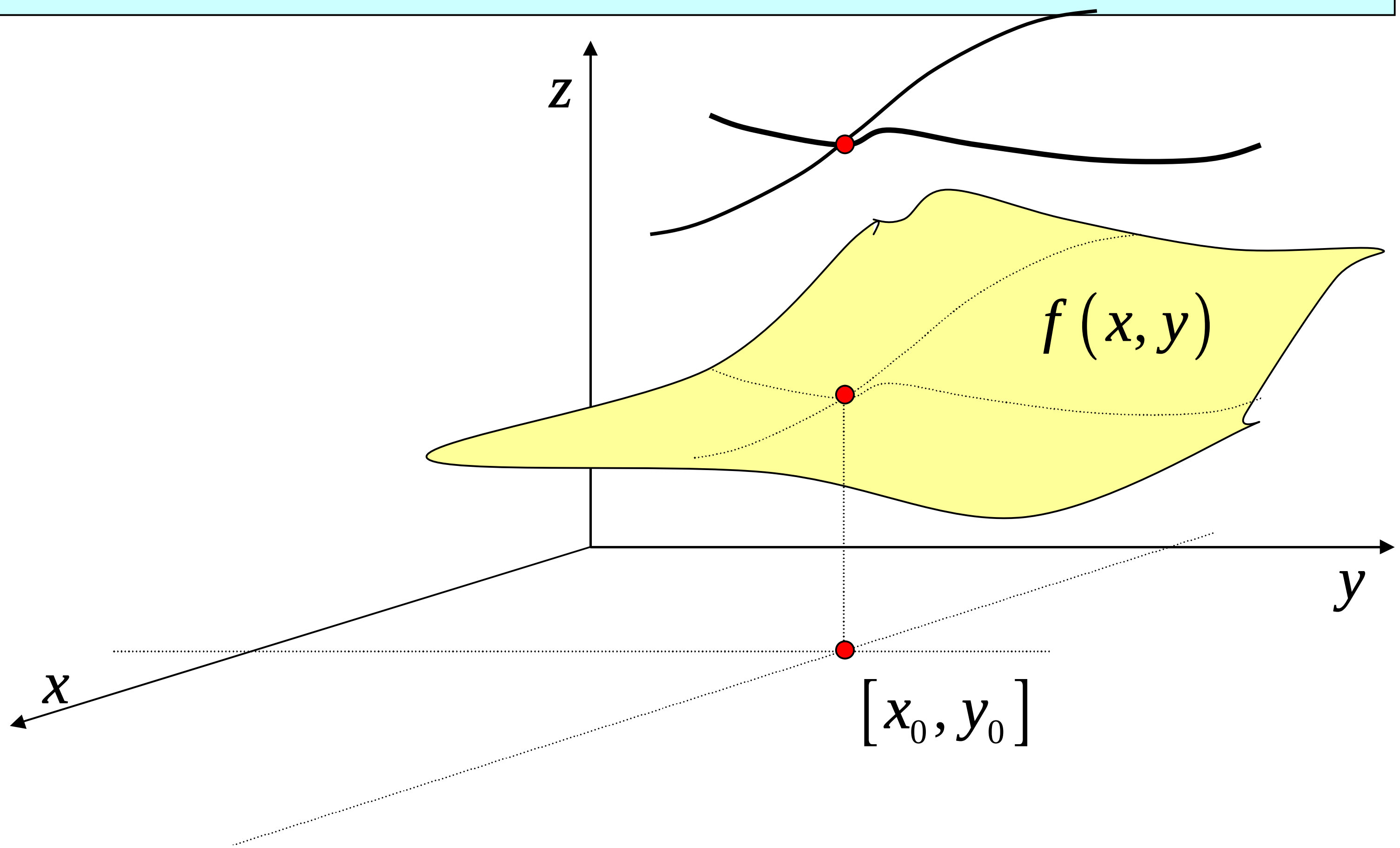
If a function of one variable is differentiable at a point, it is also continuous at this point. With functions in more variables, this implication is not so straightforward. There are functions with all partial derivatives at a point that are not continuous.



Let a function $f\left(X\right) = f\left(x_1, x_2, \ldots, x_n\right)$ be given and let A be an internal point of its domain. Let us consider a function

$df_A\left(dx_1, dx_2, \ldots, dx_n\right) = df_A\left(X\right)$ of n variables where $dx_i = x_i - a_i$

$$\boxed{df_A\left(X\right) = \frac{\partial f\left(A\right)}{\partial x_1}dx_1 + \frac{\partial f\left(A\right)}{\partial x_2}dx_2 + \cdots + \frac{\partial f\left(A\right)}{\partial x_n}dx_n}$$

If $\lim\limits_{X \to A} \dfrac{f\left(X\right) - f\left(A\right) - df_A\left(X\right)}{r\left(X, A\right)} = 0$ then the function $df_A\left(X\right)$

is called the **total differential** of $f(X)$ at A.

If a function $f(x_1, x_2, \mathbf{K}, x_n)$, at some point $A \in D(f)$, has all the partial derivatives and if al these partial derivatives are continuous functions at A, then $f(x_1, x_2, \mathbf{K}, x_n)$ has a total differential at A.

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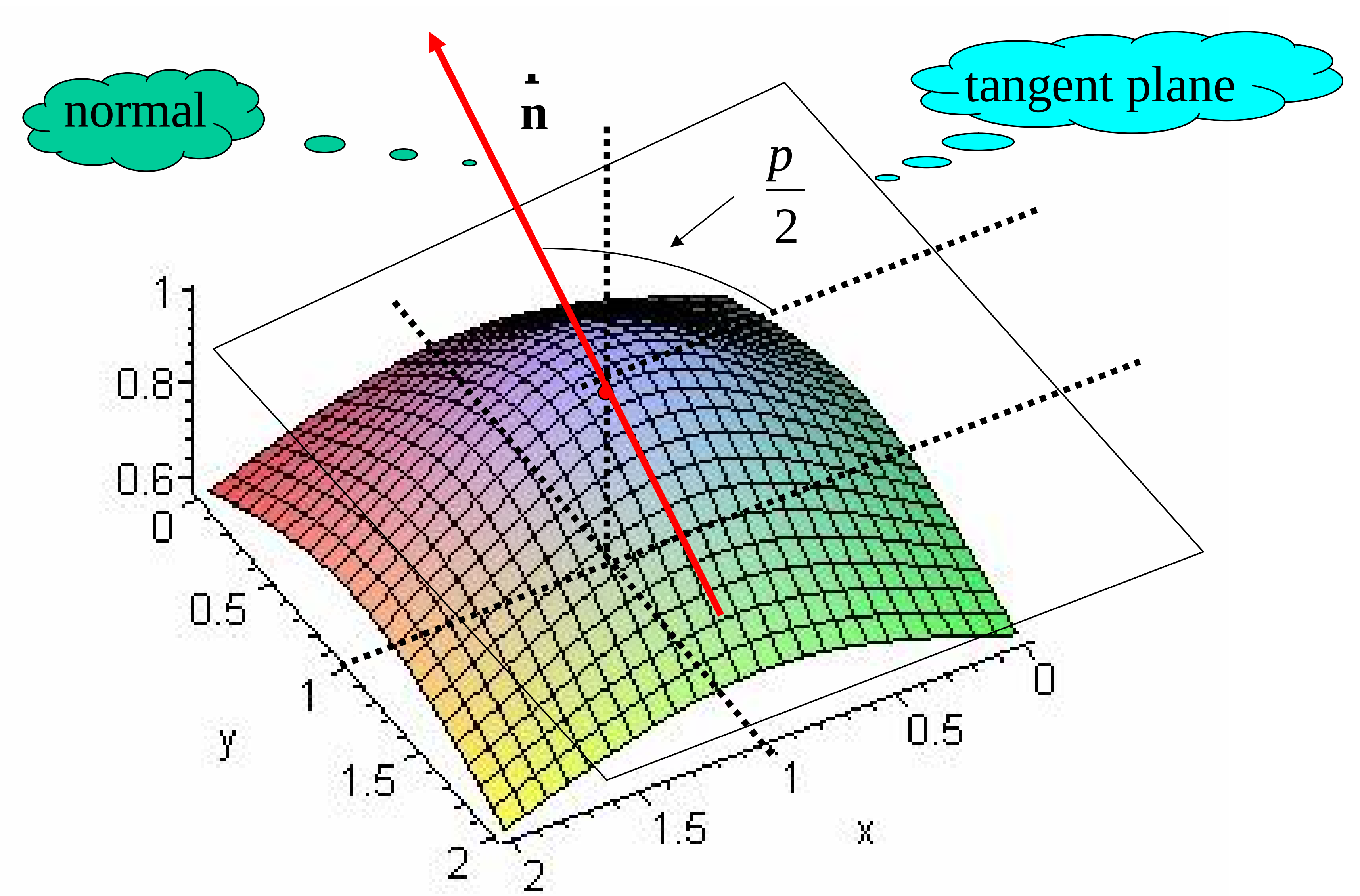
Instead od saying that a function has a total differential at a point A we can also say that it is differentiable at A.

If a function $f\left(x_1, x_2, \ldots, x_n\right)$, at some point $A \in D\left(f\right)$, has a total differential, then $f\left(x_1, x_2, \ldots, x_n\right)$ is continuous at A.

Tangent plane to the graph of a 2-function

If a function $f\left(x,y\right)$ is differentiable at a point $A=\left[a_1,a_2\right]$ its graph has a tangent plane at this point that can be defined by the formula:

$$\boxed{z-f\left(a_1,a_2\right)=\frac{\partial f\left(a_1,a_2\right)}{\partial x}\left(x-a_1\right)+\frac{\partial f\left(a_1,a_2\right)}{\partial y}\left(y-a_1\right)}$$



normal
tangent plane
n
p
2
1
0.8
0.6
0
0.5
1
y
1.5
2
2
1.5
1
x
0.5
0

Equations of the normal to the graph of a function at a point

If $f_x^{'}(a_1,a_2)\cdot f_y^{'}(a_1,a_2)\neq 0$

$A=[a_1,a_2]$

$$\frac{x-a_1}{-f_x^{'}(a_1,a_2)}=\frac{y-a_2}{-f_y^{'}(a_1,a_2)}=z-f(a_1,a_2)$$

parametric equations

$$x=a_1+t\frac{\partial f(a_1,a_2)}{\partial x}$$

$$y=a_2+t\frac{\partial f(a_1,a_2)}{\partial y}$$

$$z=f(a_1,a_2)+t$$

If we determine the total differential of a function $f\left(x_1, x_2, \ldots, x_n\right)$ at every point of $D(f)$ where it exits, we have actually defined a function in $2n$ variables:

$$df\left(x_1, x_2, \ldots, x_n, dx_1, dx_2, \ldots, dx_n\right)$$

Now we can view $dx_1, dx_2, \mathbf{K}, dx_n$ as parameters and determine the total differential of the function df with regard to the variables $x_1, x_2, \mathbf{K}, x_n$ at every point at which it exists. In this way we have defined a total differential of the total differential $df\left(x_1, x_2, \mathbf{K}, x_n\right)$ which is called a second total differential of $f\left(x_1, x_2, \mathbf{K}, x_n\right)$ and so on.

Example

For a two-function $f\left(x,y\right)$ we have

$$df\left(x,y\right)=f_x^{'}\left(x,y\right)dx+f_y^{'}\left(x,y\right)dy$$

$$\frac{\partial df\left(x,y\right)}{\partial x}=f_{xx}^{''}\left(x,y\right)dx+f_{yx}^{''}\left(x,y\right)dy$$

$$\frac{\partial df\left(x,y\right)}{\partial y}=f_{xy}^{''}\left(x,y\right)dx+f_{yy}^{''}\left(x,y\right)dy$$

$$d\left(df\left(x,y\right)\right)=d^2f\left(x,y\right)=f_{xx}^{''}dx^2+\left(f_{yx}^{''}+f_{xy}^{''}\right)dxdy+f_{yy}^{''}dy^2$$

or $d^2f\left(x,y\right)=f_{xx}^{''}dx^2+2f_{xy}^{''}dxdy+f_{yy}^{''}dy^2$ if $f_{xy}^{''}\left(x,y\right)=f_{yx}^{''}\left(x,y\right)$

To calculate the *n*-th differential of $f\left(x_1, x_2, \mathbf{K}, x_n\right)$ we can use the following formal expression

$$d^n f\left(x_1, x_2, \mathbf{K}, x_n\right) = \left(\frac{\partial}{\partial x_1} dx_1 + \frac{\partial}{\partial x_2} dx_2 + \mathbf{L} + \frac{\partial}{\partial x_n} dx_n\right)^n f\left(x_1, x_2, \mathbf{K}, x_n\right)$$

Example Calculate the third differential of $w = f(x, y, z)$

$$d^3 f(x,y,z) = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right)^3 f(x,y,z)$$

$$\left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right)^3 = \frac{3!}{3!0!0!} \frac{\partial^3}{\partial x^3} dx^3 + \frac{3!}{2!1!0!} \frac{\partial^3}{\partial x^2 \partial y} dx^2 dy +$$

$$+ \frac{3!}{2!0!1!} \frac{\partial^3}{\partial x^2 \partial z} dx^2 dz + \frac{3!}{1!2!0!} \frac{\partial^3}{\partial x \partial y^2} dx dy^2 + \frac{3!}{1!0!2!} \frac{\partial^3}{\partial x \partial z^2} dx dz^2 +$$

$$\frac{3!}{1!1!1!} \frac{\partial^3}{\partial x \partial y \partial z} dx dy dz + \frac{3!}{0!1!2!} \frac{\partial^3}{\partial y \partial z^2} dy dz^2 + \frac{3!}{0!2!1!} \frac{\partial^3}{\partial y^2 \partial z} dy^2 dz +$$

$$+ \frac{3!}{0!3!0!} \frac{\partial^3}{\partial y^3} dy^3 + \frac{3!}{0!0!3!} \frac{\partial^3}{\partial z^3} dz^3$$

$$d^3 f(x,y,z) = \frac{\partial^3 f(x,y,z)}{\partial x^3} dx^3 + 3\frac{\partial^3 f(x,y,z)}{\partial x^2 \partial y} dx^2 dy +$$

$$+3\frac{\partial^3 f(x,y,z)}{\partial x^2 \partial z} dx^2 dz + 3\frac{\partial^3 f(x,y,z)}{\partial x \partial y^2} dx dy^2 + 3\frac{\partial^3 f(x,y,z)}{\partial x \partial z^2} dx dz^2 +$$

$$6\frac{\partial^3 f(x,y,z)}{\partial x \partial y \partial z} dx dy dz + 3\frac{\partial^3 f(x,y,z)}{\partial y \partial z^2} dy dz^2 + 3\frac{\partial^3 f(x,y,z)}{\partial y^2 \partial z} dy^2 dz +$$

$$+\frac{\partial^3 f(x,y,z)}{\partial y^3} dy^3 + \frac{\partial^3 f(x,y,z)}{\partial z^3} dz^3$$

Using a total differential find the value of $2.1^{3.2}$

We will calculate the total differential of the function

$z = x^y$ at $x = 2$ and $y = 3$ and then substitute $dx = 0.1$, $dy = 0.2$

$$2.1^{3.2} = z(2.1, 3.2) \approx z(2,3) + dz\,|_{dx=0.1, dy=0.2}$$

$$\frac{\partial z}{\partial x} = yx^{y-1} \Rightarrow 3 \cdot 2^2 = 12$$

$$\frac{\partial z}{\partial y} = x^y \ln x \Rightarrow 2^3 \ln 2 \approx 5.54518$$

$$2.1^{3.2} \approx 8 + 12(0.1) + 5.54518(0.2) = 10.30904$$