Implicitly defined function

Let $f\left(x_1, x_2, \mathbf{K}, x_n\right)$ and $g\left(x_1, x_2, \mathbf{K}, x_n, y\right)$ be two functions such

that, for every $\left(x_1, x_2, \mathbf{K}, x_n\right) \in D\left(f\right)$, we have

$$\left(x_1, x_2, \mathbf{K}, x_n, f\left(x_1, x_2, \mathbf{K}, x_n\right)\right) \in D\left(g\right)$$

and

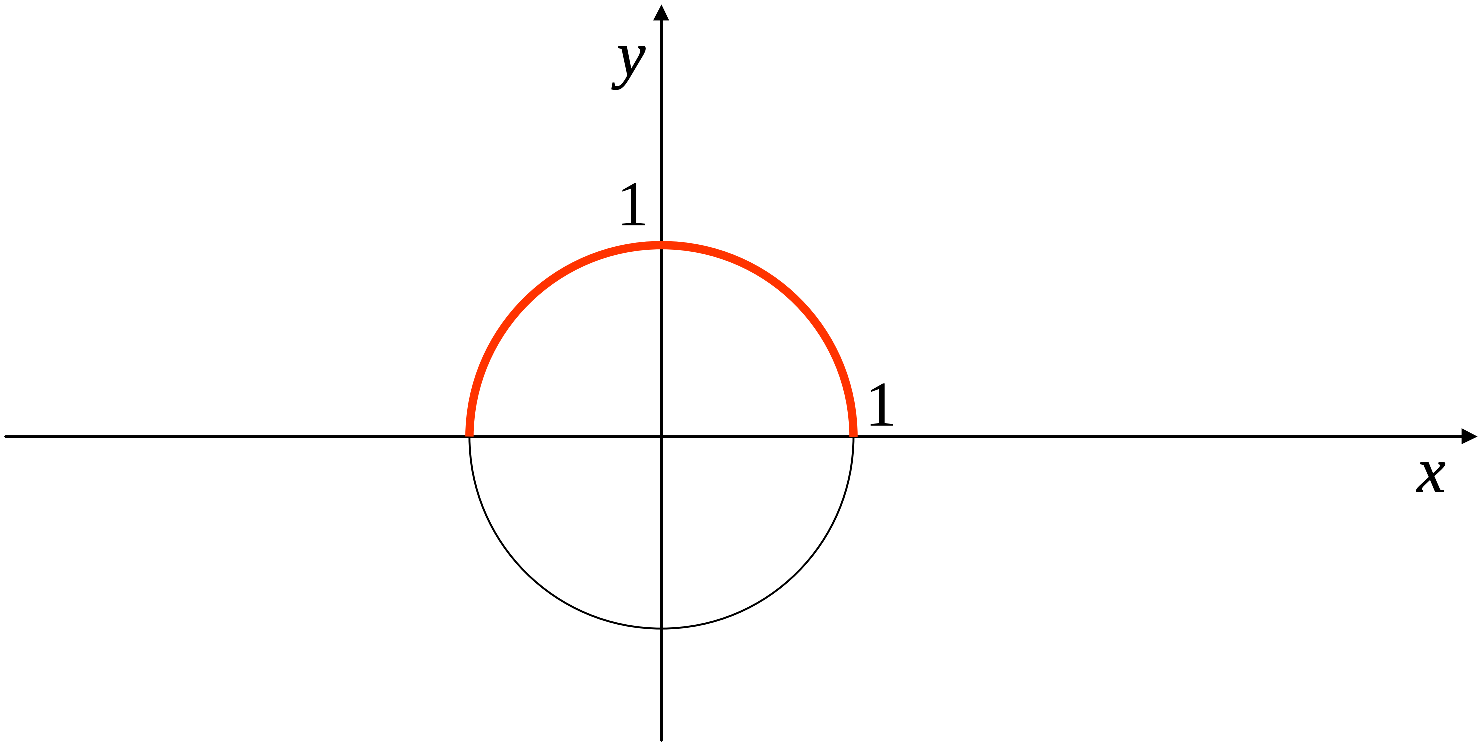
$$\boxed{g\left(x_1, x_2, \mathbf{K}, x_n, f\left(x_1, x_2, \mathbf{K}, x_n\right)\right) = 0}$$

Then we say that the function $y = f\left(x_1, x_2, \mathbf{K}, x_n\right)$ **is defined implicitly** by the equation $g\left(x_1, x_2, \mathbf{K}, x_n, y\right) = 0$

Example

The equation $x^2 + y^2 - 1 = 0$ defines for $y \geq 0$ implicitly

the function $y = \sqrt{1 - x^2}$



Does every equation

$$g\left(x_1, x_2, \mathbf{K}, x_n, y\right) = 0$$

define implicitly a function

$$y = f\left(x_1, x_2, \mathbf{K}, x_n\right)$$

?

Let the function $g\left(x_1, x_2, \mathbf{K}, x_n, y\right)$ have **<u>continuous partial</u>**

<u>derivatives in a neighbourhood</u> of a point $\left(a_1, a_2, \mathbf{K}, a_n, b\right)$

and let $g\left(a_1, a_2, \mathbf{K}, a_n, b\right) = 0,\ g_y^{'}\left(a_1, a_2, \mathbf{K}, a_n, b\right) \neq 0$

Then there exists a continuous function $f\left(x_1, x_2, \mathbf{K}, x_n\right)$ in a

neighbourhood of $\left(a_1, a_2, \mathbf{K}, a_n\right)$ given implicitly by the equation

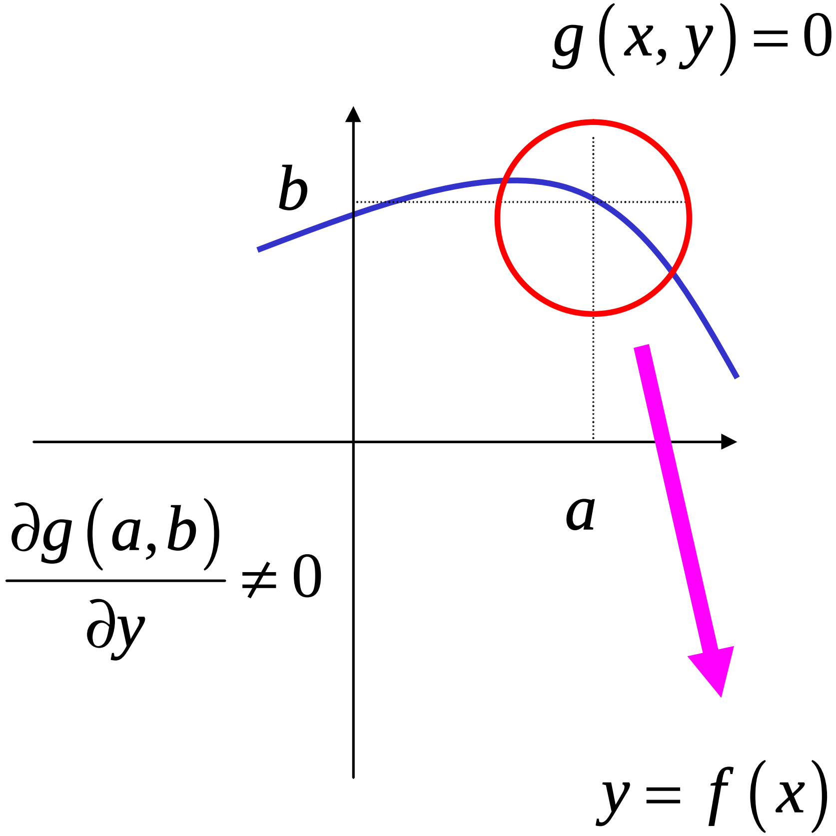
$$g\left(x_1, x_2, \mathbf{K}, x_n, y\right) = 0$$

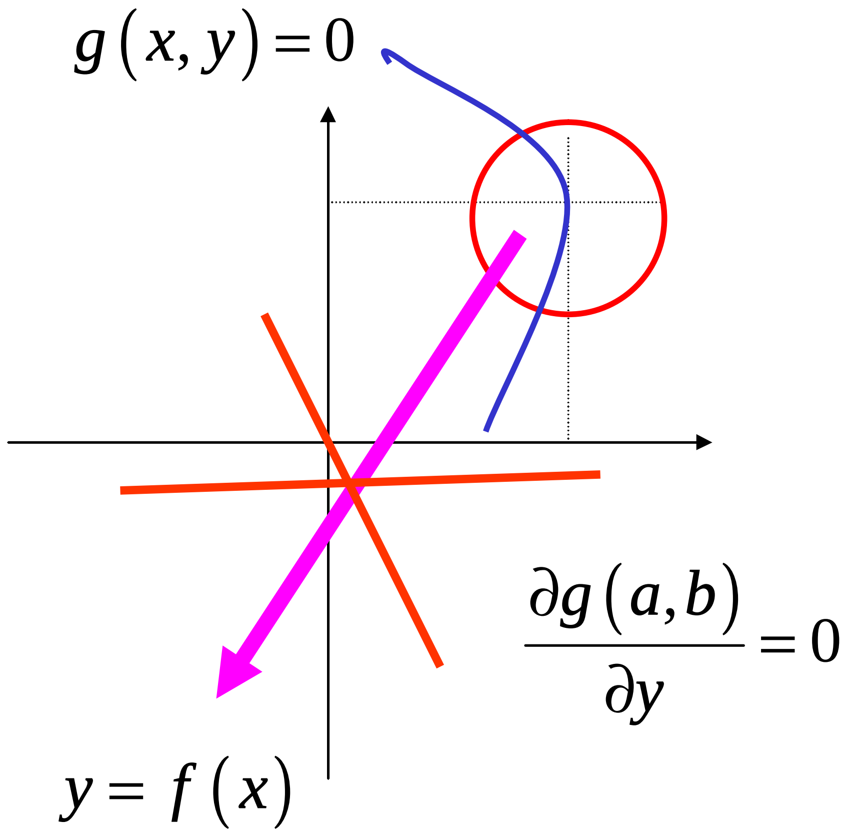
and

the function $f\left(x_1, x_2, \mathbf{K}, x_n\right)$ has partial derivatives at the point

$\left(a_1, a_2, \mathbf{K}, a_n\right)$ given by the formula

$$\frac{\partial f\left(a_1, a_2, \mathbf{K}, a_n\right)}{\partial x_i} = -\frac{\dfrac{\partial g\left(a_1, a_2, \mathbf{K}, a_n, b\right)}{\partial x_i}}{\dfrac{\partial g\left(a_1, a_2, \mathbf{K}, a_n, b\right)}{\partial y}}$$


$g(x,y)=0$
b
a
$\frac{\partial g(a,b)}{\partial y} \neq 0$
$y=f(x)$


$g(x,y)=0$
$\frac{\partial g(a,b)}{\partial y} = 0$
$y=f(x)$

Example

$x^2 - 2y^2 + 3z^2 - yz + y = 0$ Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $[x, y] = [1,1]$

$z = {}^1\!/_3$

$$x^2 - 2y^2 + 3z^2 - yz + y = 0 \quad \Rightarrow d\left(x^2 - 2y^2 + 3z^2 - yz + y\right) = 0$$

$$2xdx - (4y - z + 1)dy + (6z - y)dz = 0$$

$$dz = -\frac{2x}{6z - y}dx + \frac{4y + z - 1}{6z - y}dy$$

$$\frac{\partial z}{\partial x} = -2, \frac{\partial z}{\partial y} = \frac{10}{3}$$

Higher order partial derivatives of implicitly defined functions

When calculating a partial derivative of, say, order k, we differentiate a suitable previously calculated partial derivative of order k - 1, substituting previously calculated values whenever necessary.

Example

Find $\dfrac{\partial^2 z}{\partial x^2}$ of the function defined as above

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{-2(6z - y) + 2x \cdot 6\dfrac{\partial z}{\partial x}}{(6z - y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-2(2-1) + 2 \cdot 1 \cdot 6(-2)}{(1)^2} = -26$$