Another sufficient condition of local minima/maxima

Let a function $f\left(x_1, x_2, \ldots, x_n\right)$ have continuous second order partial derivatives at a point $A \in E_n$ and let $\dfrac{\partial f\left(A\right)}{\partial x_i} = 0, i = 1, 2, \ldots, n$

If $d^2 f\left(A\right) < 0$ for all $dx_1, dx_2, \ldots, dx_n$, then $f\left(x_1, x_2, \ldots, x_n\right)$ has a local maximum at A.

If $d^2 f\left(A\right) > 0$ for all $dx_1, dx_2, \ldots, dx_n$, then $f\left(x_1, x_2, \ldots, x_n\right)$ has a local minimum at A.

Relative maximum

Let $f\left(x_1, x_2, \ldots, x_n\right)$ be defined on a set $M \subseteq E_n$ and let a set U be defined by the constraints

$$\boxed{\begin{aligned} g_1\left(x_1, x_2, \ldots, x_n\right) &= 0 \\ g_2\left(x_1, x_2, \ldots, x_n\right) &= 0 \\ &\vdots \\ g_k\left(x_1, x_2, \ldots, x_n\right) &= 0 \end{aligned}}$$

If, for a point $A \in U$, there is a neighbourhood $N(A, \delta)$ such that $f\left(X\right) \leq f\left(A\right)$ for $X \in N\left(A, \delta\right) \cap U$, we say that $f\left(x_1, x_2, \ldots, x_n\right)$ has a local maximum at A relative to U.

Relative minimum

Let $f\left(x_1, x_2, \ldots, x_n\right)$ be defined on a set $M \subseteq E_n$ and let a set U be defined by the constraints

$$\boxed{\begin{aligned} g_1\left(x_1, x_2, \ldots, x_n\right) &= 0 \\ g_2\left(x_1, x_2, \ldots, x_n\right) &= 0 \\ &\vdots \\ g_k\left(x_1, x_2, \ldots, x_n\right) &= 0 \end{aligned}}$$

If, for a point $A \in U$, there is a neighbourhood $N(A, \delta)$ such that $f\left(X\right) \geq f\left(A\right)$ for $X \in N\left(A, \delta\right) \cap U$, we say that $f\left(x_1, x_2, \ldots, x_n\right)$ has a local minimum at A relative to U.

Method of Lagrange multipliers

Let U be given by the set of constraints

$$g_i\left(x_1, x_2, \ldots, x_n\right) = 0, i = 1, 2, \ldots, k$$

We are to find local minima or maxima of $f\left(x_1, x_2, \ldots, x_n\right)$

relative to U.

First we define $L\left(x_1, x_2, \ldots, x_n\right)$ called a Lagrange function

as
$$\boxed{L\left(x_1, x_2, \ldots, x_n\right) = f\left(x_1, x_2, \ldots, x_n\right) + \sum_{n=1}^{k} \lambda_i g_i\left(x_1, x_2, \ldots, x_n\right)}$$

The numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ are called Lagrange multipliers.

We are now looking for points where the first partial derivatives of $L(x_1, x_2, \ldots, x_n)$ are equal to zero (stationary points) while satisfying the constraints. This leads to the following system of equations:

$$\boxed{\begin{array}{l}
\dfrac{\partial f(X)}{\partial x_1} + \lambda_1 \dfrac{\partial g_1(X)}{\partial x_1} + \cdots + \lambda_k \dfrac{\partial g_k(X)}{\partial x_1} = 0 \\
\qquad\qquad\qquad\qquad \vdots \\
\dfrac{\partial f(X)}{\partial x_n} + \lambda_1 \dfrac{\partial g_1(X)}{\partial x_n} + \cdots + \lambda_k \dfrac{\partial g_k(X)}{\partial x_n} = 0 \\
g_1(X) = g_2(X) = \cdots = g_k(X) = 0
\end{array}} \qquad (*)$$

We have $n + k$ equations for $n + k$ unknowns

$$x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_k$$

By solving (*), we find a set $M \subseteq E_n$ of candidates

for local maxima/minima.

Finally, we test each $X \in M$ whether

$d^2L(X) < 0$ (maximum) or $d^2L(X) > 0$ (minimum)

using the fact that the first differentials of the constraints have

to be zero at X:

$$\boxed{\begin{aligned}
&\frac{\partial g_1(X)}{\partial x_1}dx_1 + \frac{\partial g_1(X)}{\partial x_2}dx_2 + \cdots + \frac{\partial g_1(X)}{\partial x_n}dx_n = 0\\
&\frac{\partial g_2(X)}{\partial x_1}dx_1 + \frac{\partial g_2(X)}{\partial x_2}dx_2 + \cdots + \frac{\partial g_2(X)}{\partial x_n}dx_n = 0\\
&\qquad\qquad\qquad\qquad\vdots\\
&\frac{\partial g_k(X)}{\partial x_1}dx_1 + \frac{\partial g_k(X)}{\partial x_2}dx_2 + \cdots + \frac{\partial g_k(X)}{\partial x_n}dx_n = 0
\end{aligned}}$$

Example

$z = xy - x + y - 1$ Constraint: $x + y = 1$

$$L(x, y) = xy - x + y + \lambda(x + y - 1)$$

$$\frac{\partial L}{\partial x} = y - 1 + \lambda, \quad \frac{\partial L}{\partial y} = x + 1 + \lambda \qquad \frac{\partial^2 L}{\partial x^2} = 0, \quad \frac{\partial^2 L}{\partial y^2} = 0, \quad \frac{\partial^2 L}{\partial x \partial y} = 1$$

$$\begin{array}{ccccccc} x & + & y & & & = & 1 \\ & & y & + & \lambda & = & 1 \\ x & & & + & \lambda & = & -1 \end{array}$$

$$d^2 L = dxdy$$

$$x + y = 1 \Rightarrow dx + dy = 0 \Rightarrow dy = -dx$$

$$d^2 L = -dxdx < 0$$

$$\text{Solution}: \quad x = \frac{-1}{2}, y = \frac{3}{2}, \lambda = \frac{-1}{2}$$

Relative maximum at $\left[\frac{-1}{2}; \frac{3}{2}\right]$

Example

$$w = x^2 + y^2 + z^2 \qquad \text{Constraints:} \qquad \begin{aligned} x + y - 3z + 7 &= 0 \\ x - y + \ \ z - 3 &= 0 \end{aligned}$$

$$L(x, y, z) = x^2 + y^2 + z^2 + \lambda_1 (x + y - 3z + 7) + \lambda_2 (x - y + z - 3)$$

$$\frac{\partial L}{\partial x} = 2x + \lambda_1 + \lambda_2 = 0, \frac{\partial L}{\partial y} = 2y + \lambda_1 - \lambda_2 = 0, \frac{\partial L}{\partial z} = 2x - 3\lambda_1 + \lambda_2 = 0$$

$$\text{Solution}: \quad x = 0, y = -1, z = 2, \lambda_1 = 1, \lambda_2 = -1$$

$$\frac{\partial^2 L}{\partial x^2} = \frac{\partial^2 L}{\partial y^2} = \frac{\partial^2 L}{\partial z^2} = 2, \quad \frac{\partial^2 L}{\partial x \partial y} = \frac{\partial^2 L}{\partial x \partial z} = \frac{\partial^2 L}{\partial y \partial z} = 0$$

$$d^2 L = 2(dx^2 + dy^2 + dz^2) > 0$$

Relative minimum at [0, - 1,2]