Let $f(x_1, x_2, \ldots, x_n)$ be a function with $D(f) \subseteq E_n$

If, for a point $A = [a_1, a_2, \ldots, a_n] \in D(f)$ we have

$$f(a_1, a_2, \ldots, a_n) \leq f(x_1, x_2, \ldots, x_n)$$

for all $X = [x_1, x_2, \ldots, x_n] \in D(f)$ then we say that f reaches

its **minimum** at *A*. Similarly, if

$$f(a_1, a_2, \ldots, a_n) \geq f(x_1, x_2, \ldots, x_n)$$

for any $X = [x_1, x_2, \ldots, x_n] \in D(f)$, then we say that f reaches

its **maximum** at *A*.

Let $f\left(x_1, x_2, \ldots, x_n\right)$ be a function with $D\left(f\right) = M \subseteq E_n$

If $A \in M$ is a minimum/maximum of $f\left(x_1, x_2, \ldots, x_n\right)$

then

- f has a local maximum/minimum at A

  or

- f has a local maximum/minimum at A relative to the boundary of M

When establishing maxima and minima of a function

$f(x_1, x_2, \dots, x_n)$ on M, we use the following method

- find all the local maxima/minima of $f(x_1, x_2, \dots, x_n)$
- find all the local maxima/minima of $f(x_1, x_2, \dots, x_n)$ relative to ∂M
- among such maxima/minima, chose the greatest and the lowest values

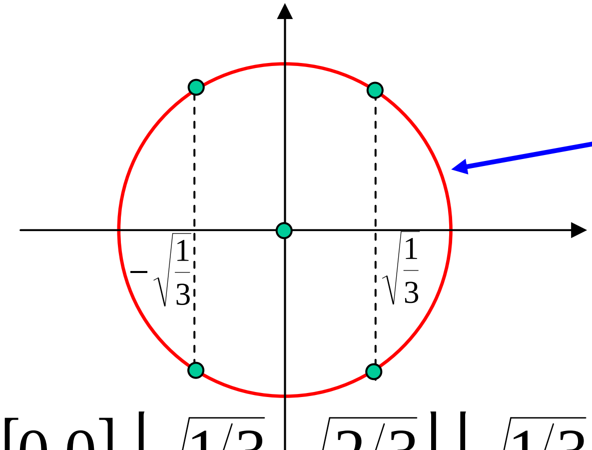
Note that, when establishing local minima/maxima of a function $f(x_1,x_2,...,x_n)$, if there are only a few stationary points with the first partial derivatives equal to zero, it need not be necessary to determine for each of the candidates whether it is really a local or relative maximum or minimum, but rather put it to the final test of substituting it into $f(x_1, x_2, ..., x_n)$.

Example

Find the maxima/minima of $z = x^2 y$ on $x^2 + y^2 \le 1$

$$\frac{\partial z}{\partial x} = 2xy = 0, \frac{\partial z}{\partial y} = x^2 = 0 \Rightarrow x = y = 0$$

$$z' = 1 - 3y^2 = 0 \Rightarrow y_{1,2} = \pm\sqrt{\frac{1}{3}}, x = \pm\sqrt{\frac{2}{3}}$$



$$z = y(1 - y^2)$$

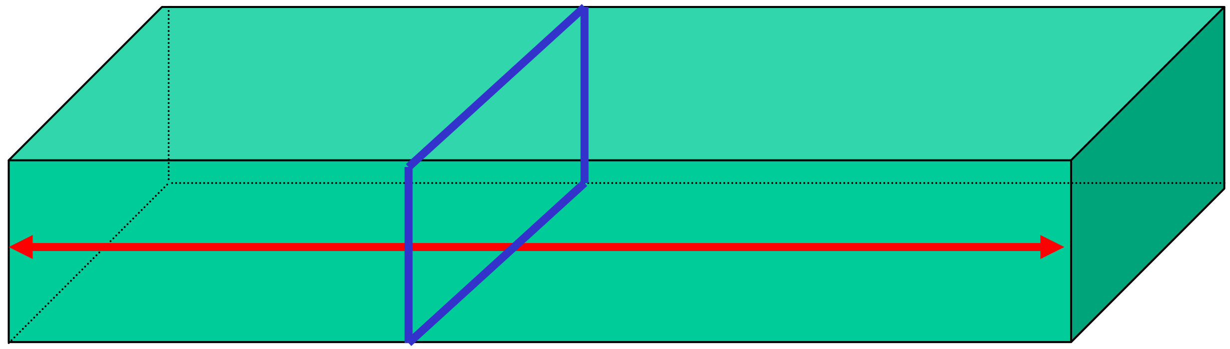
Candidates for maxima/minima:

$$[0,0], \left[\sqrt{1/3}, \sqrt{2/3}\right], \left[\sqrt{1/3}, -\sqrt{2/3}\right], \left[-\sqrt{1/3}, \sqrt{2/3}\right], \left[-\sqrt{1/3}, -\sqrt{2/3}\right],$$

0	$\sqrt{2}/3\sqrt{3}$	$-\sqrt{2}/3\sqrt{3}$	$\sqrt{2}/3\sqrt{3}$	$-\sqrt{2}/3\sqrt{3}$
	MAX	MIN	MAX	MIN

Example

The sum of the length and girth (perimeter of the cross section) of packages carried by parcel post cannot exceed 108 inches. Find the dimension of the rectangular package of largest volume that may be sent by parcel post.



length – l, widht – w, height – h, girth = $2(w+h)+l$

Find the greatest value of $V = whl$ with $2(w+h)+l \leq 108$ (*)

$$L(w,h,l) = whl - \lambda(2w + 2h + l - 108)$$

$$\frac{\partial L}{\partial w} = hl - 2\lambda = 0, \quad \frac{\partial L}{\partial h} = wl - 2\lambda = 0, \quad \frac{\partial L}{\partial l} = wh - \lambda = 0$$

This means that $h = t, w = t, l = 2t$.

Substituting into $2(w+h)+l = 108$ yields: $6t = 108 \rightarrow t = 18$

Disregarding obviously $w = h = l = 0$ as the only possible local maximum, the only remaining candidate seems to be $w = 18$, $h = 18$, $l = 36$.

Let us calculate the second differential of L:

Clearly, $\frac{\partial^2 L}{\partial w^2} = \frac{\partial^2 L}{\partial h^2} = \frac{\partial^2 L}{\partial l^2} = 0.$ Next $\frac{\partial L}{\partial w \partial h} = l, \frac{\partial L}{\partial w \partial l} = h, \frac{\partial L}{\partial l \partial h} = w.$

Thus at [18,18,36] we have $d^2 L = 36 dwdh + 18 dwdl + 18 dlhd.$

But $2w + 2h + l = 0$ means that $2dw + 2dh + dl = 0$ and so

$$d^2 L = 4\left\{(2dw + dh + dl)^2 - \left(4dw^2 + dh^2 + dl^2\right)\right\}$$
$$= -16dw^2 - 4dh^2 - 4dl^2 < 0$$

This means that at [18,18,36] V reaches its greatest value.