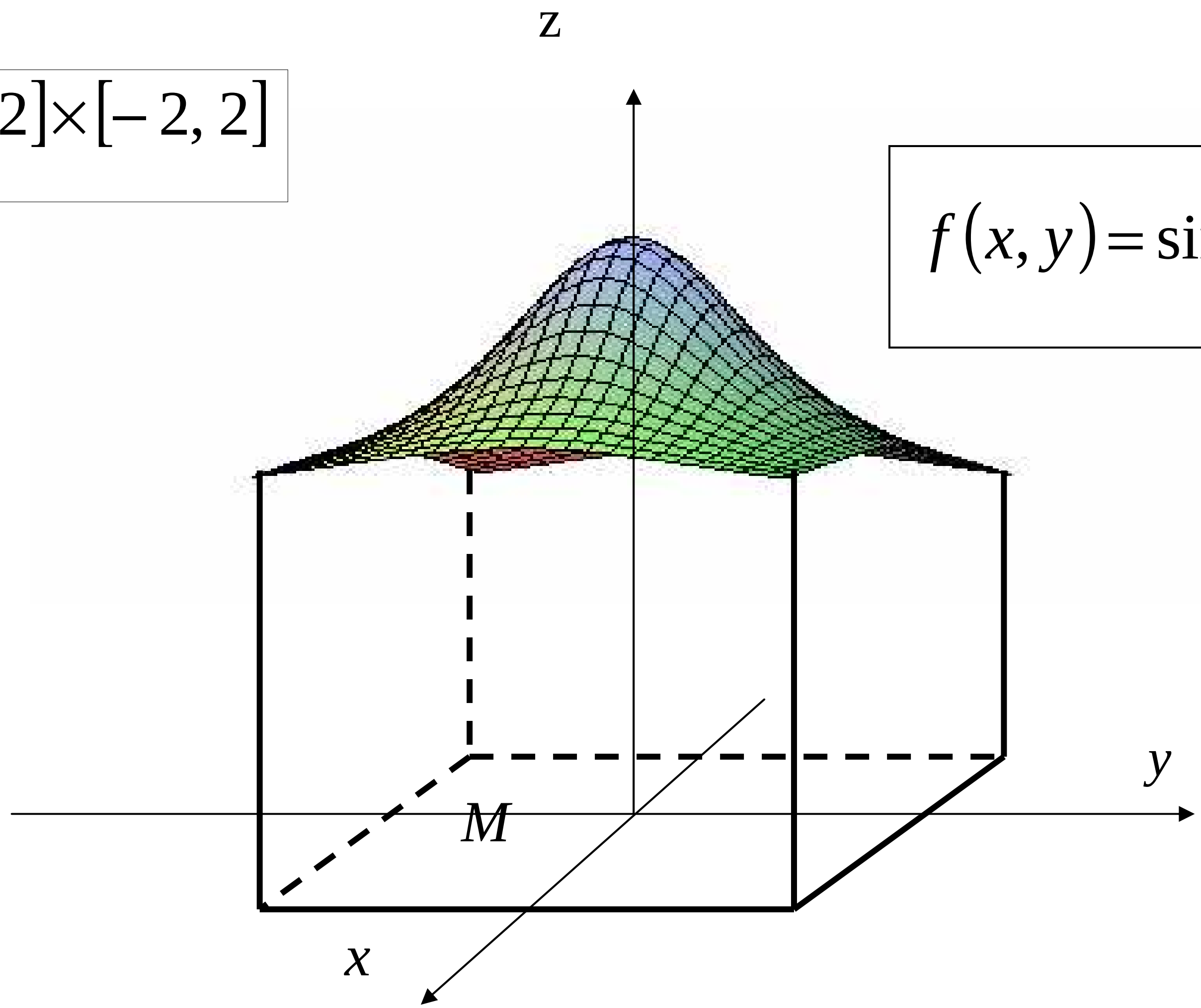
DOUBLE INTEGRAL

Perhaps the simplest motivation for the introduction of a double integral is the need to establish the volume of a solid bounded by a continuous function $f(x, y)$ defined over a bounded closed planar area *M*.

$$M = [-2, 2] \times [-2, 2]$$

$$f(x, y) = \sin \frac{1}{1 + x^2 + y^2}$$



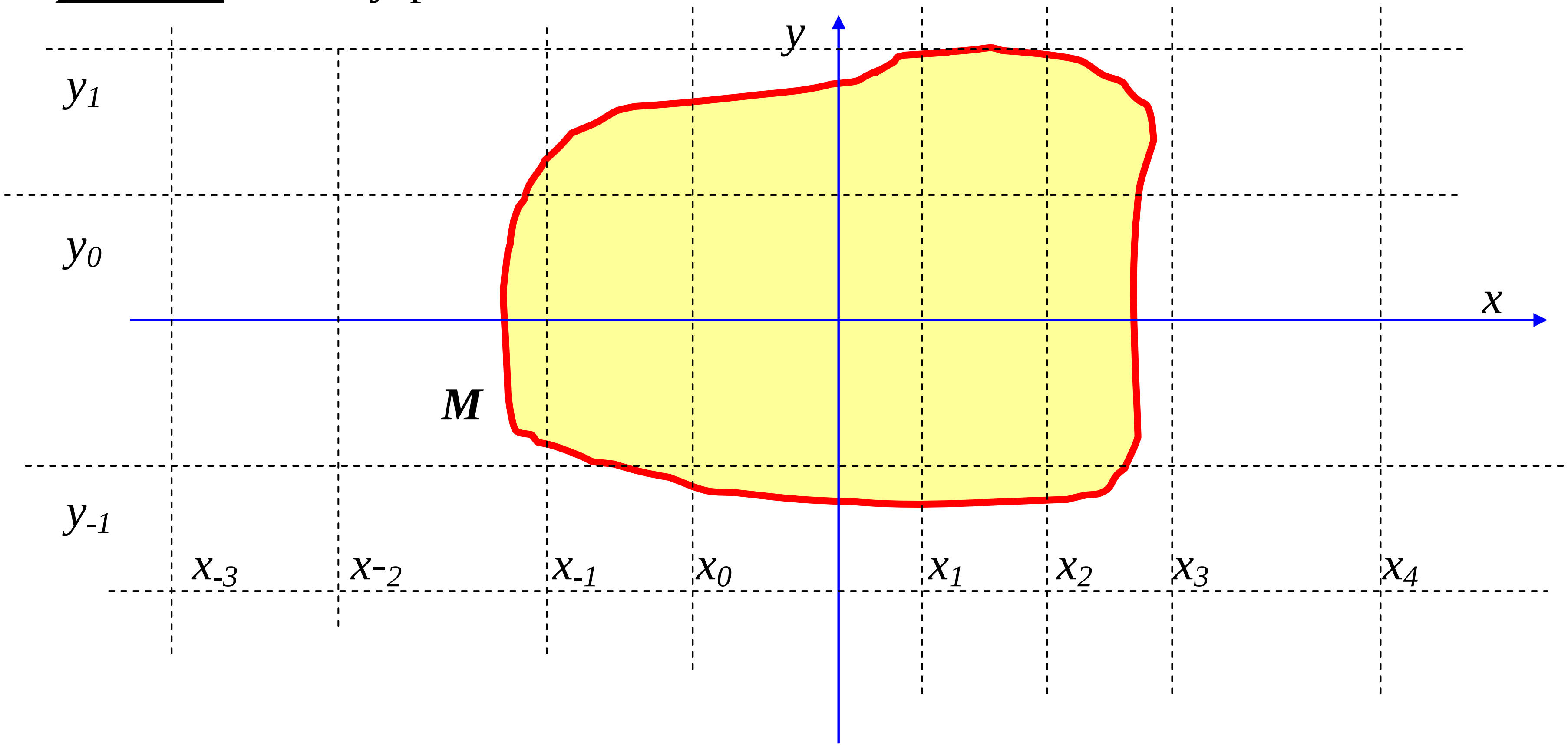
PARTITION OF A PLANAR AREA

Let … x_{-3} , x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, … and … y_{-3} , y_{-2}, y_{-1}, y_0, y_1, y_2, y_3, … be sequences of real numbers such that $\lim_{n\to\infty} x_{-n} = -\infty$, $\lim_{n\to\infty} x_n = \infty$ $\lim_{n\to\infty} y_{-n} = -\infty$, and $\lim_{n\to\infty} y_n = \infty$.

We say that the perpendicular grid formed by the straight lines

$$x = x_n,\ n = \ldots -3,-2,-1,0,1,2,3\ldots, \quad y = y_n,\ n = \ldots -3,-2,-1,0,1,2,3\ldots,$$

is a ***partition*** of the *xy*-plane. Let *M* denote a bounded closed area.



THE INTEGRAL SUMS

Let $f(x, y)$ be a function defined on *M*. Let us denote $\Delta x_i = x_{i+1} - x_i$, $\Delta y_j = y_{j+1} - x_j$, and define the

lower integral sum

$$I_L = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0}} \sum_i \sum_j f\left(x_{m_i}, y_{m_j}\right) \Delta x_i \Delta y_j$$

where $f\left(x_{m_i}, y_{m_j}\right) = \inf_{(x,y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]} f(x, y)$, and

the ***upper integral sum***

$$I_U = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0}} \sum_i \sum_j f\left(x_{M_i}, y_{M_j}\right) \Delta x_i \Delta y_j$$

where $f\left(x_{M_i}, y_{M_j}\right) = \sup_{(x,y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]} f(x, y)$,

- The sums are taken for those values of i and j for which the square $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ has a non-empty intersection with M.

- The symbol $\inf\limits_{(x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]} f(x,y)$ denotes the ***infimum*** or the ***greatest lower bound*** (GLB) of the function $f(x,y)$ over the rectangular area $[x_i, y_i] \times [y_j, y_{j+1}]$

- The symbol $\sup\limits_{(x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]} f(x,y)$ denotes the ***supremum*** or the ***least upper bound*** (LUB) of the function $f(x,y)$ over the rectangular area $[x_i, y_i] \times [y_j, y_{j+1}]$

DOUBLE INTEGRAL

If both the integral sums I_L and I_U of $f(x, y)$ exist with $I_L = I_U$, we say that the function $f(x, y)$ is integrable over M or with respect to M and put

$$I_L = I_U = \iint\limits_M f(x, y)\, dx\, dy.$$

The expression $\iint\limits_M f(x, y)\, dx\, dy$ is called the ***double integral of*** $f(x, y)$ ***with respect to*** *M*.

ADDITIVE PROPERTIES OF DOUBLE INTEGRAL

If $c_1, c_2, \ldots c_n$ are constants and $f_1(x,y),\ldots, f_n(x,y)$ are integrable functions with respect to an area *M*, then $c_1 f_1(x,y)+\ldots+c_n f_n(x,y)$ is an integrable function with respect to *M* and

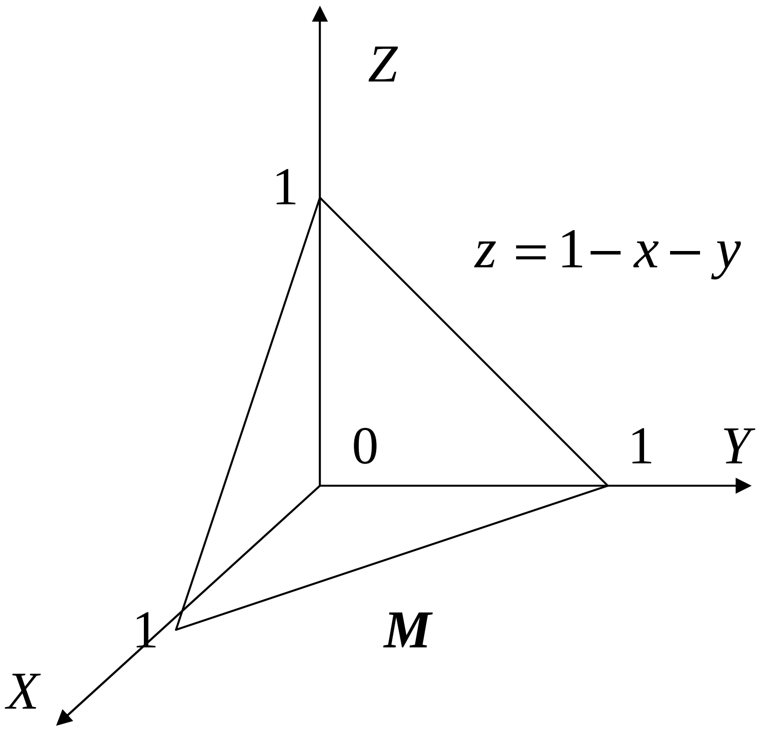
$$\iint_M c_1 f_1(x,y)+\ldots+c_n f_n(x,y)dxdy = c_1\iint_M f_1(x,y)dxdy+\ldots+c_n\iint_M f_n(x,y)dxdy$$

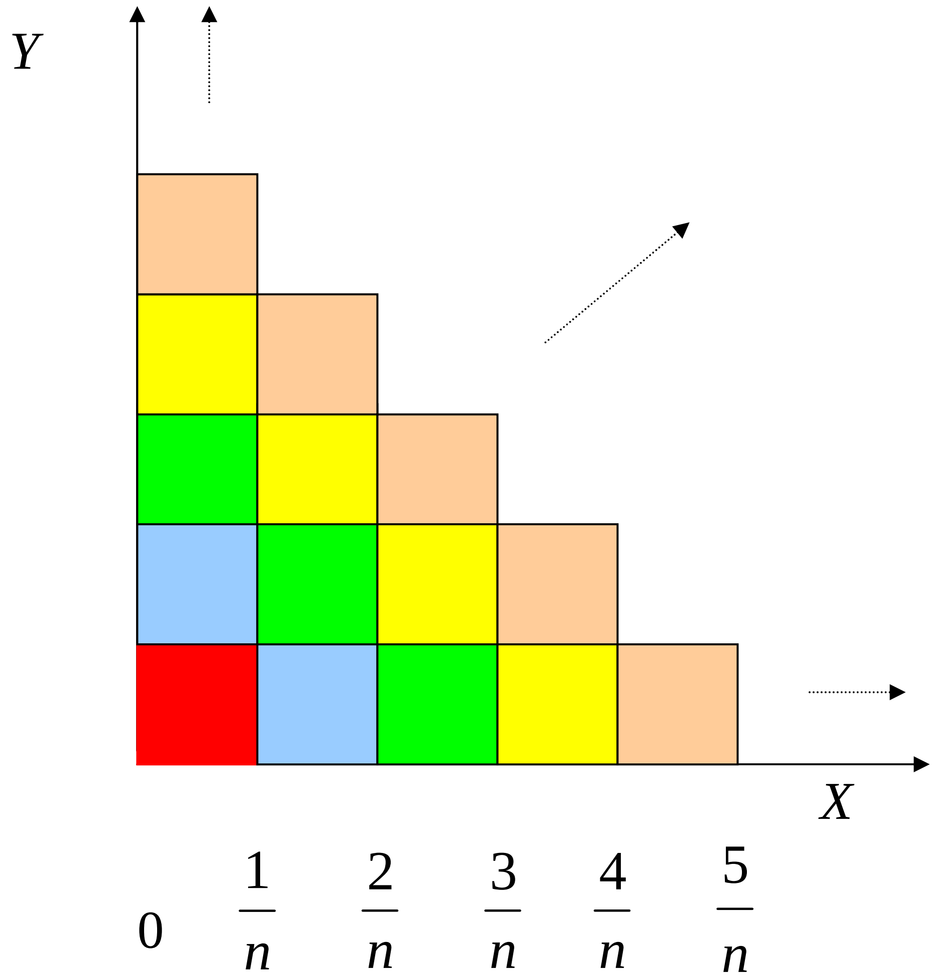
If $M_1,\ldots,M_n$ are pair-wise disjunct bounded closed planar areas, $M = M_1+\cdots+M_n$ and $f(x,y)$ a function integrable with respect to all $M_i, i=1,\ldots,n$. Then $f(x,y)$ is integrable with respect to *M* and

$$\iint_M f(x,y)dxdy = \iint_{M_1} f(x,y)dxdy+\cdots+\iint_{M_n} f(x,y)dxdy$$

EXAMPLE

Prove that the double integral of $f(x,y)=1-x-y$ with regard to an area M given by the inequalities $x\geq 0,\ y\geq 0,\ x+y\leq 1$ exists and calculate its value.





The figure on the right shows the grid that we will use to partition *M*. It is not difficult to establish that the lub and glb of $f(x, y)$ over the red area is $1-\frac{0}{n}$ and $1-\frac{1}{n}$ respectively, similarly, for the blue, green, yellow, and brown areas we obtain the following values of suprema and infima respectively $1-\frac{1}{n}$, $1-\frac{2}{n}$; $1-\frac{2}{n}$, $1-\frac{3}{n}$; $1-\frac{3}{n}$, $1-\frac{4}{n}$ and $1-\frac{4}{n}$, $1-\frac{5}{n}$. Clearly, it takes $1+2+\cdots+n=\frac{n(n+1)}{2}$ squares to cover *M* each square having an area of $\frac{1}{n^2}$. This gives us the following expressions used in the definitions of integral sums for $f(x, y)$:

$$I_L(n)=\frac{1}{n^2}\left[1\left(1-\frac{1}{n}\right)+2\left(1-\frac{2}{n}\right)+3\left(1-\frac{3}{n}\right)+\cdots+n\left(1-\frac{n}{n}\right)\right]$$

$$I_U(n)=\frac{1}{n^2}\left[1\left(1-\frac{0}{n}\right)+2\left(1-\frac{1}{n}\right)+3\left(1-\frac{2}{n}\right)+\cdots+n\left(1-\frac{n-1}{n}\right)\right]$$

Since, clearly, $\lim_{n\to\infty} I_L(n) = I_L$ and $\lim_{n\to\infty} I_U(n) = I_U$, all we have to do now is calculate these limits.

$$I_L(n) = \frac{1}{n^2}\left[1\left(1-\frac{1}{n}\right)+2\left(1-\frac{2}{n}\right)+3\left(1-\frac{3}{n}\right)+\cdots+n\left(1-\frac{n}{n}\right)\right] =$$

$$= \frac{1}{n^2}\left[1+2+\cdots n - \frac{1}{n}\left(1^2+2^2+\cdots+n^2\right)\right] = \frac{1}{n^2}\left[\frac{(n+1)n}{2} - \frac{1}{n}\frac{n\left(2n^2+3n+1\right)}{6}\right] =$$

$$\frac{n+1}{2n} - \frac{2n^2+3n+1}{6n^2}$$ and so $$\lim_{n\to\infty} I_L(n) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

$$I_U(n) = \frac{1}{n^2}\left[1\left(1-\frac{0}{n}\right)+2\left(1-\frac{1}{n}\right)+3\left(1-\frac{2}{n}\right)+\cdots+n\left(1-\frac{n-1}{n}\right)\right] =$$

$$= \frac{1}{n^2}\left[1+2+\cdots n - \frac{1}{n}\left((1)(0)+(2)(1)+\cdots+(n)(n-1)\right)\right] = \frac{1}{n^2}\left[\frac{(n+1)n}{2} - \frac{1}{n}\frac{n^3-n}{3}\right] =$$

$$\frac{n+1}{2n} - \frac{n^2-1}{3n^2}$$ and again we have $$\lim_{n\to\infty} I_L(n) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

NOTE 1

The sum $S_n^2 = 1^2 + 2^2 + \cdots + n^2$ can be calculated as follows. Denote by S_n the sum $1 + 2 + \cdots + n$. Thus we have $S_n = \dfrac{n(n+1)}{2}$. Let us now use the well-known formula $(n+1)^3 = 3n^2 + 3n + 1$ to write

$$\begin{aligned} 2^3 - 1^3 &= (3)1^2 + (3)(1) + 1 \\ 3^3 - 2^3 &= (3)2^2 + (3)(2) + 1 \\ \vdots \quad & \quad \vdots \qquad \quad \vdots \\ (n+1)^3 - n^3 &= 3n^2 + 3n + 1 \end{aligned}$$

Adding up these equations yields

$(n+1)^3 - n^3 = 3S_n^2 + 3S_n + n$ and, after some simplification, we finally get $S_n^2 = \dfrac{n(2n^2 + 3n + 1)}{6}$.

NOTE 2

As far as the sum $S_{n(n-1)} = (1)(2) + (2)(3) + \cdots + (n-1)n$ is concerned, note that, if we put $T_n(x) = 1 + x + x^2 + \cdots + x^n$, we have $T''_n(1) = S_{n(n-1)}$. Since $T_n(x)$ is a geometric series, we can write $T_n(x) = \dfrac{1-x^{n+1}}{1-x}$ for $x \neq 1$ and $T_n(1) = \lim\limits_{x\to 1} \dfrac{1-x^n}{1-x}$.

Now $T''_n(1) = \lim\limits_{x\to 1} \dfrac{dx^2}{dx^2} \dfrac{1-x^{n+1}}{1-x}$. Differentiating twice, we obtain

$$\frac{dx^2}{dx^2}\frac{1-x^{n+1}}{1-x} = \frac{n(n-1)x^{n+1} - 2(n+1)(n-1)x^n n(n+1)x^{n-1} - 2}{(x-1)^3}$$

and, using the l' Hospital rule, we have

$$\lim_{x\to 1} \frac{dx^2}{dx^2}\frac{1-x^{n+1}}{1-x} = \lim_{x\to 1} \frac{(n+1)n(n-1)x^n - 2(n+1)n(n-1)x^{n-1} + (n+1)n(n-1)x^{n-2}}{3(x-1)^2} =$$

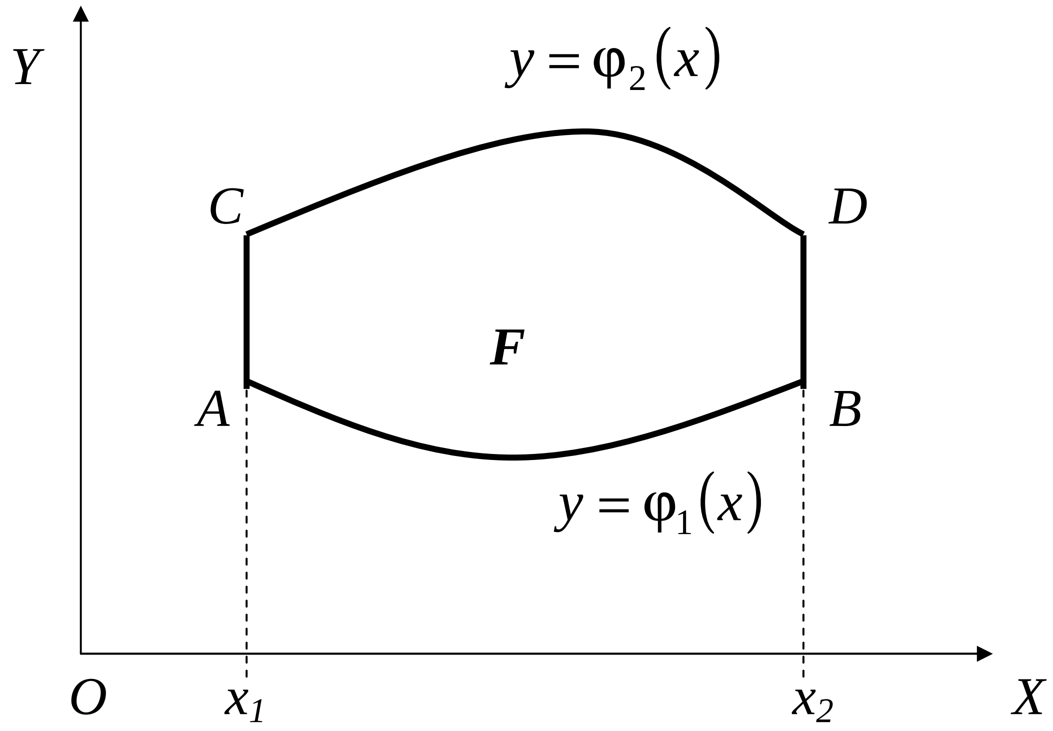
$$= \lim_{x\to 1}(n+1)n(n-1)x^{n-2} \lim_{x\to 1} \frac{x^2 - 2x + 1}{3(x-1)^2} = \frac{n^3 - n}{3}$$

Even such a relatively simple example as the one we have calculated shows that it is very difficult to calculate a double integral using the definition. For more complex functions it becomes almost impossible.

Fortunately, most of the planar areas occurring in engineering problems over which a double integral is to be calculated can be broken down into some elementary figures. For these figures, the double integral can be calculated using a method based on Fubini's theorem. Owing to the additive properties of double integral, it suffices to add up the double integrals over the elementary figures.

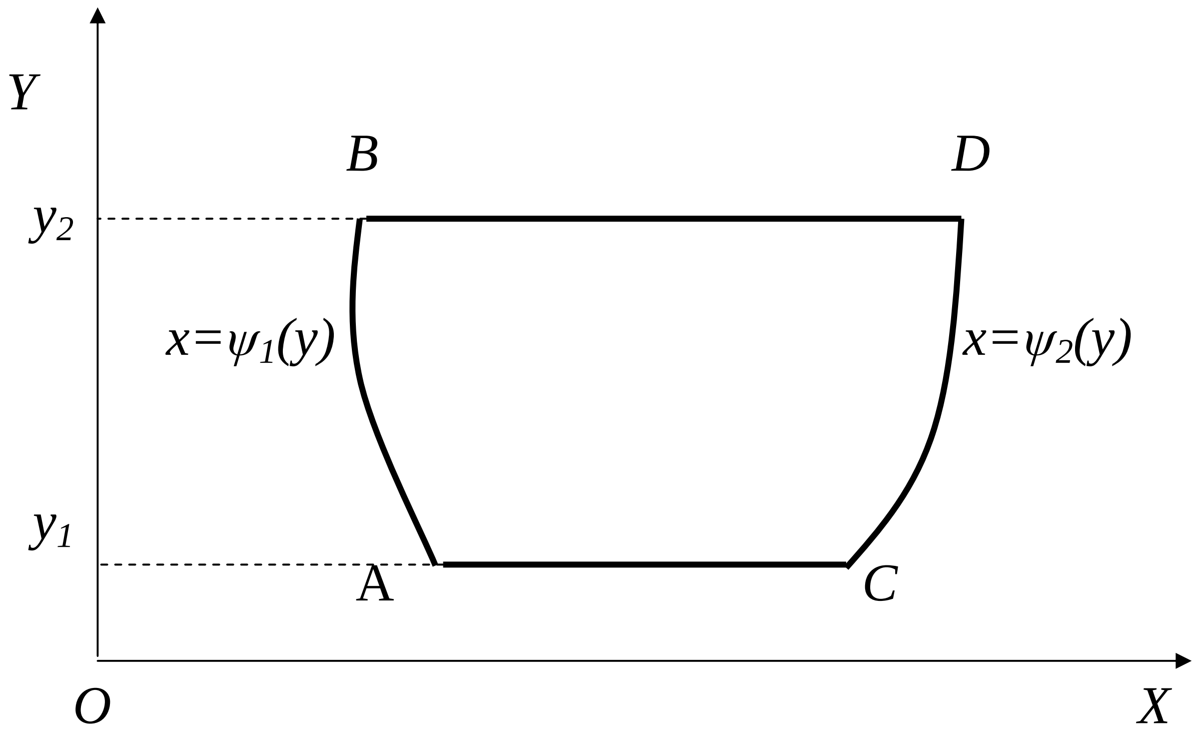
THE *φ(x)*-FIGURE

Let $\varphi_1(x), \varphi_2(x)$ be functions continuous for $x_1 \le x \le x_2$ with $\varphi_1(x) < \varphi_2(x)$. We will define a planar figure *F*, which we will call a *φ(x)*-figure, bounded on the left and right by the straight lines $x = x_1$, $x = x_2$ and, above and below, by the curves $y = \varphi_1(x)$, $y = \varphi_2(x)$.



THE *ψ(y)*-FIGURE

Let $\psi_1(y), \psi_2(y)$ be functions continuous for $y_1 \le y \le y_2$ with $\psi_1(y) < \psi_2(y)$. We will define a planar figure *G*, which we will call a *ψ(y)*-figure, bounded above and below by the straight lines $y = y_1$, $y = y_2$ and, on the left and right, by the curves $x = \psi_1(y)$, $x = \psi_2(y)$.



FUBINI'S THEOREM

Let F be a *φ(x)*-figure and G a *ψ(y)*-figure. Let $f(x,y)$ be a function integrable with respect to F and $g(x,y)$ a function integrable with respect to G. Then we can write

$$\iint_F f(x,y)\,dx\,dy = \int_{x_1}^{x_2}\left\{\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right\}dx \qquad (1)$$

$$\iint_G g(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left\{\int_{\psi_1(y)}^{\psi_2(y)} g(x,y)\,dx\right\}dy \qquad (2)$$

The expressions on the right-hand sides of (1) and (2) are called ***<u>iterated integrals</u>*** and sometimes we use the following notation

$$\iint_F f(x,y)\,dx\,dy = \int_{x_1}^{x_2} dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \qquad (1')$$

$$\iint_G g(x,y)\,dx\,dy = \int_{y_1}^{y_2} dy \int_{\psi_1(y)}^{\psi_2(y)} g(x,y)\,dx \qquad (2')$$

OTHER PROPERTIES OF DOUBLE INTEGRAL

Let $f(x, y)$ be a function integrable over a bounded closed area M and let $a \le f(x, y) \le b$ for $(x, y) \in M$. It can be easily proved using the definition that

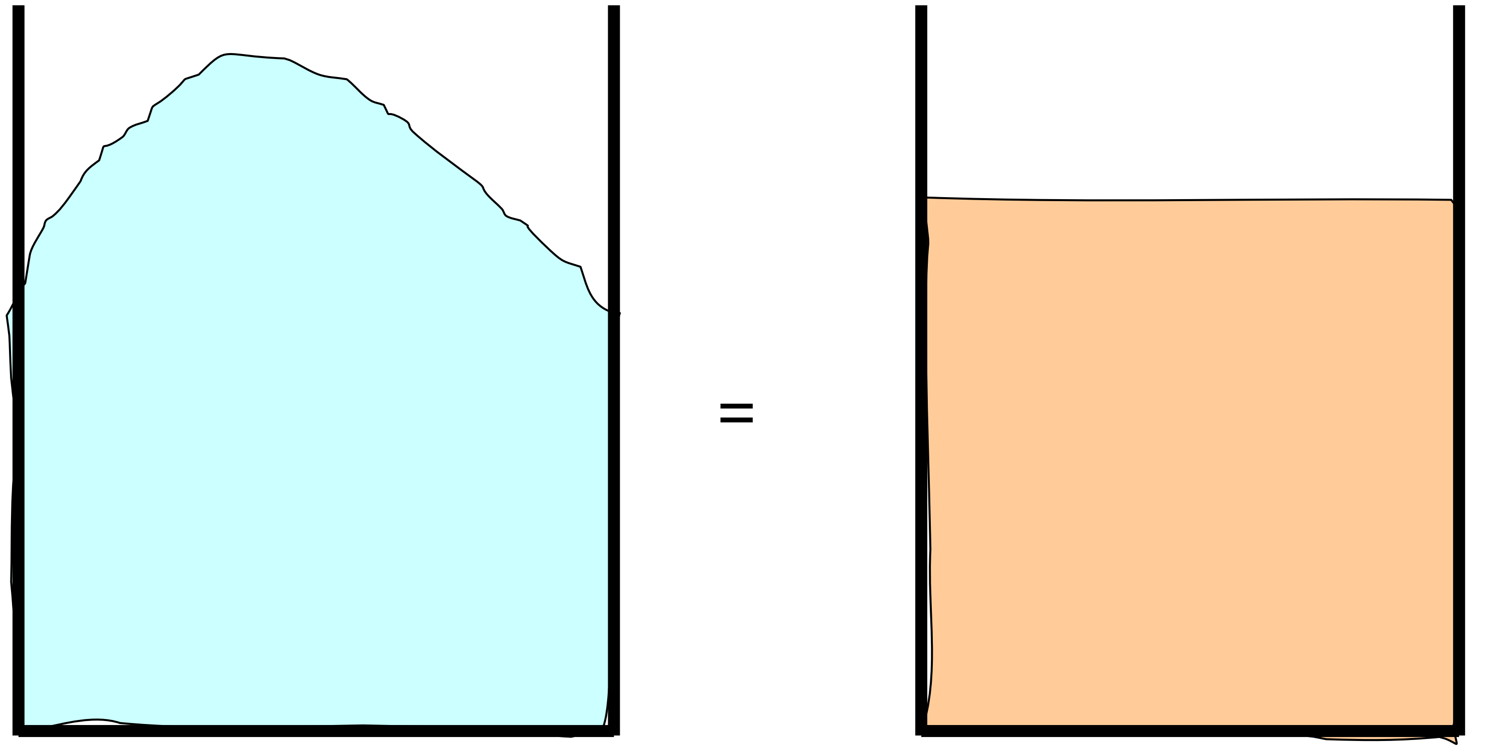
$$a|M| \le \iint_M f(x, y)\, dx\, dy \le b|M|$$

where $|M|$ denotes the area of M.

There exists a number c such that $a \le c \le b$ and

$$\iint_M f(x, y)\, dx\, dy = c|M|$$

This picture demonstrates the last property



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