DOUBLE INTEGRAL OVER RECTANGULAR AREAS

Clearly, any rectangle $R=[a,b]\times[c,d]$ is both a $\varphi(x)$ and a $\psi(y)$ figure and integration with respect to this area is extremely simple using iterated integrals.

$$\iint_R f(x,y)\,dx\,dy=\int_a^b dx\int_c^d f(x,y)\,dy=\int_c^d dy\int_a^b f(x,y)\,dx$$

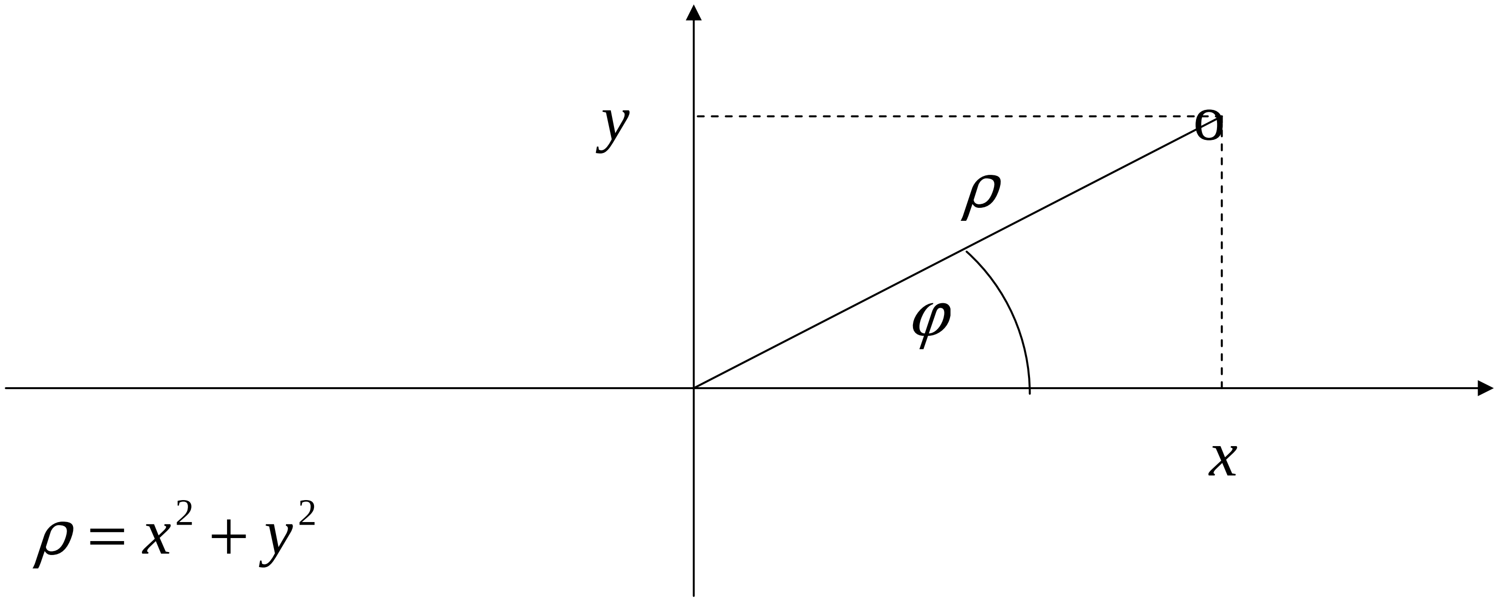
If, moreover, we can write $f(x,y)=f_1(x)f_2(y)$, we have

$$\iint_R f(x,y)\,dx\,dy=\int_a^b f_1(x)\,dx\int_c^d f_2(y)\,dy$$

POLAR COORDINATES

Some planar figures are better expressed using polar coordinates.

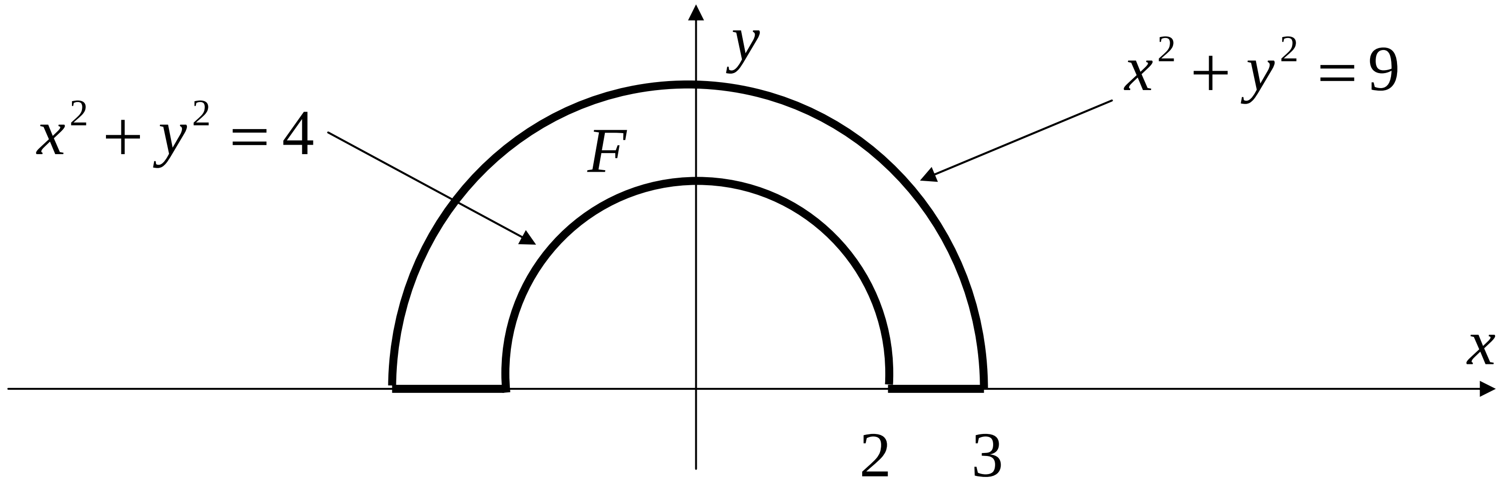
$$x = \rho \cos \varphi \qquad y = \rho \sin \varphi$$



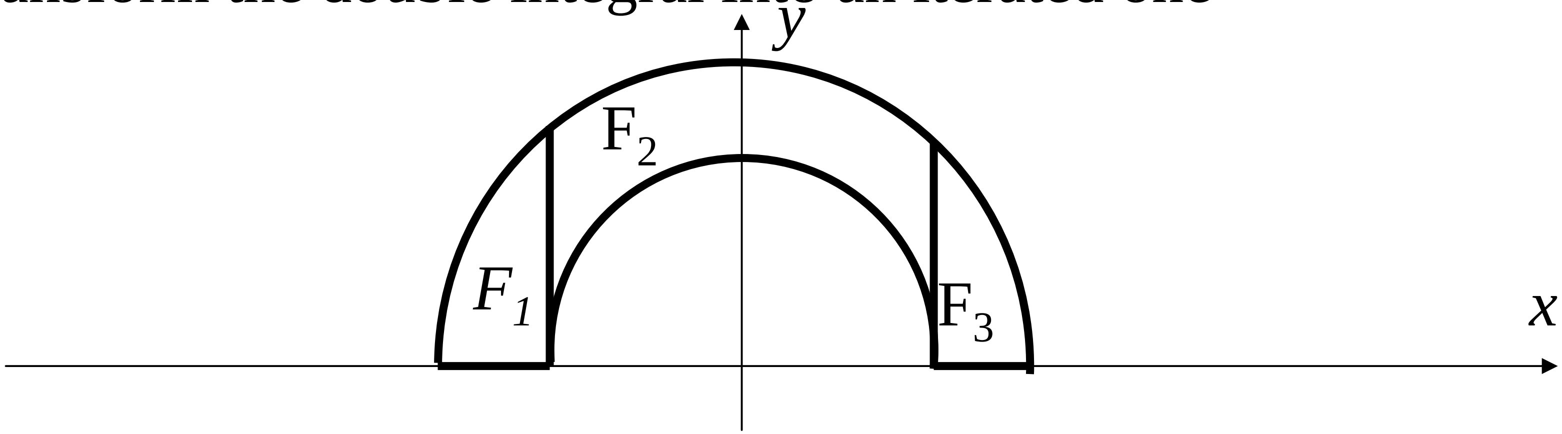
$$\rho = x^2 + y^2$$

$$\varphi = \arcsin \frac{y}{x^2 + y^2} \quad \wedge \quad \varphi = \arccos \frac{x}{x^2 + y^2}$$

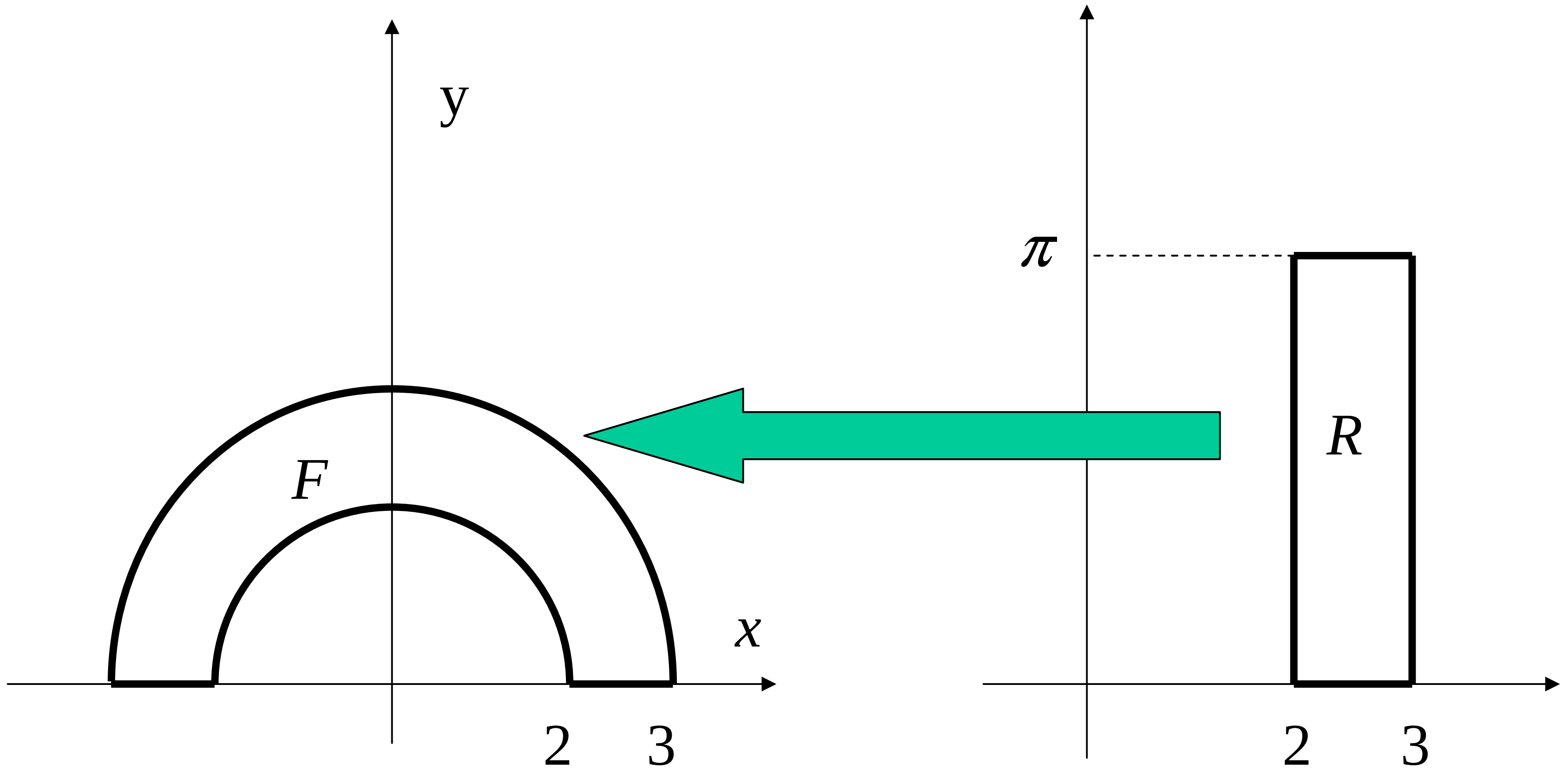
For example to calculate a double integral over the following figure



we would have to partition the figure into three to be able to transform the double integral into an iterated one



However, in the polar co-ordinate system, the same figure could be expressed as an ordinate set expressed in the polar co-ordinates $[\rho, \varphi]$ such that $2 \le \rho \le 3$ and $0 \le \varphi \ge \pi$ which is a single rectangle



CHANGE OF COORDINATES

Could we use this transformation when calculating the double integral of a function $f(x,y)$? In other words, could we substitute $\rho\cos\varphi$ for x and $\rho\sin\varphi$ for y and integrate $f(x,y)$ with respect to *R ?* The point that has to be taken into consideration is the following. Whereas the surface area of *F is* $\frac{5}{2}\pi$ that of *R is* π and, moreover, in terms of curvature, whereas *R* has a constant one, that is none, with *F* the curvature increases as ρ grows. In fact, to set things right, we would have to write $\iint\limits_F f(x,y)dxdy = \iint\limits_R f\left(\rho\cos\varphi, \rho\sin\varphi\right)\rho\, d\rho d\varphi$

Why exactly in this form will be explained in what follows on a more general basis.

TRANSFORMATIONS IN R^2

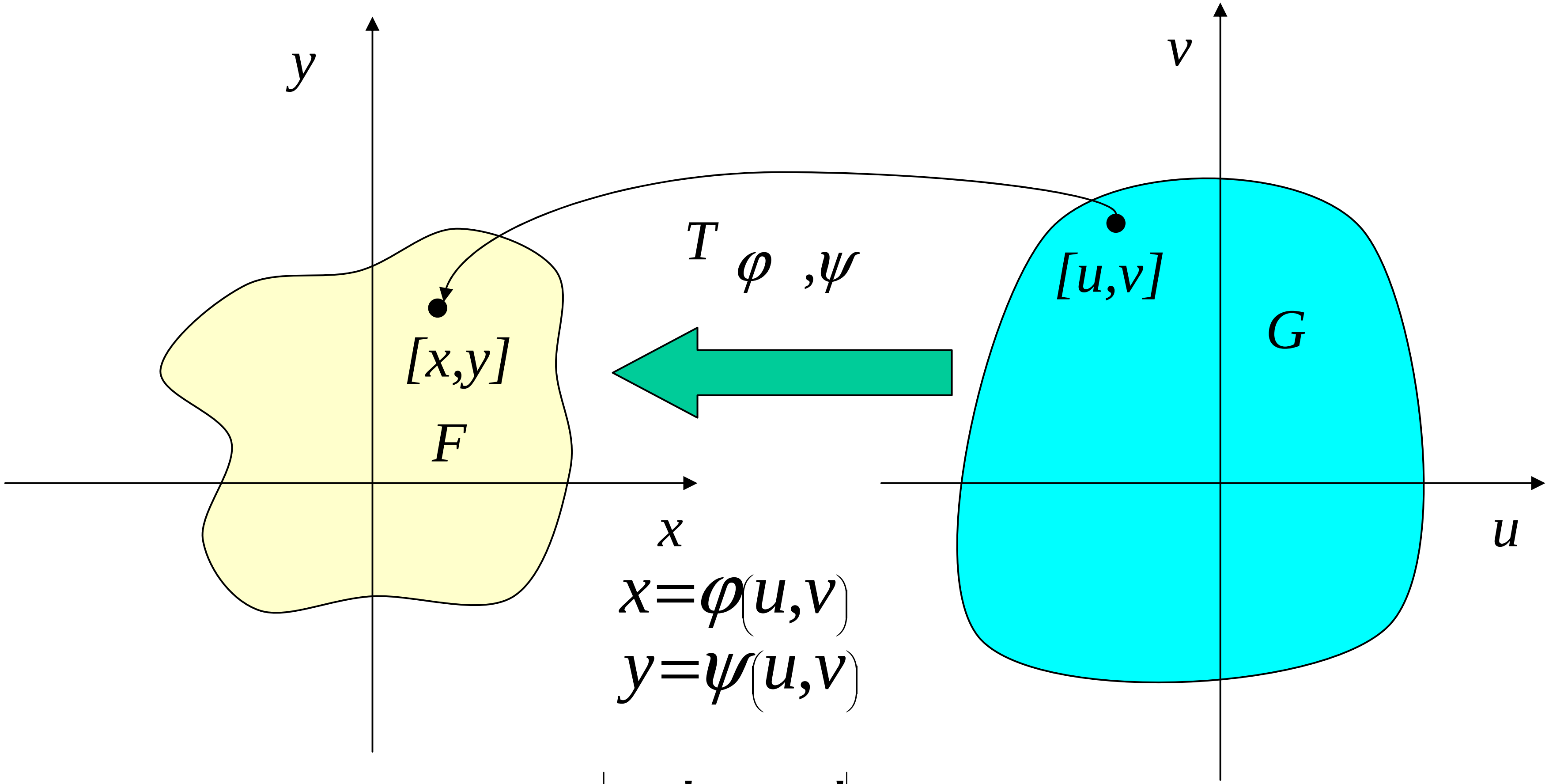
Let F, G be bounded closed areas in $\boldsymbol{R^2}$ and let $\varphi(u,v), \psi(u,v)$ be functions of two variables u,v, smooth on G, that is, with partial derivatives $\varphi_u'(u,v), \varphi_v'(u,v), \psi_u'(u,v), \psi_v'(u,v)$ continuous on G.

Put $J(u,v) = \begin{vmatrix} \varphi_u'(u,v) & \varphi_v'(u,v) \\ \psi_u'(u,v) & \psi_v'(u,v) \end{vmatrix}$ and for every $[u,v] \in G$ put

$$T_{\varphi,\psi}[u,v] = [x,y] = [\varphi(u,v), \psi(u,v)]$$

If $T_{\varphi,\psi}(G) = F$ and $J(u,v) \neq 0,\ [u,v] \in G$ we say that $T_{\varphi,\psi}$ is a regular transformation of G onto F.

$J(u,v)$ is called the ***Jacobian determinant*** of $T_{\varphi,\psi}$



$$x=\varphi(u,v)$$
$$y=\psi(u,v)$$

$$J(u,v)=\begin{vmatrix}\varphi_u' & \varphi_v' \\ \psi_u' & \psi_v'\end{vmatrix}=\varphi_u'\psi_v'-\varphi_v'\psi_u'$$

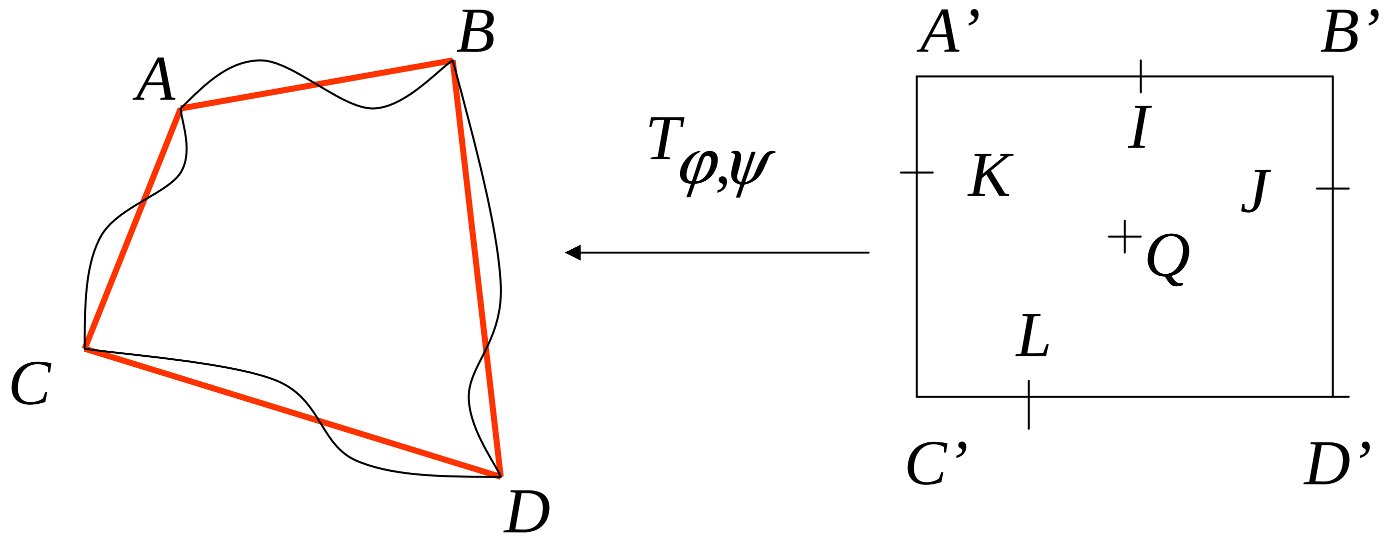
CHANGE OF VARIABLES IN DOUBLE INTEGRAL

Let $f(x,y)$ be a function integrable over an area *F* and let $T_{\varphi,\psi}:(\varphi(u,v),\psi(u,v))$ be a regular transformation of an area *G* with the *J(u,v)* as its Jacobian determinant. Then the function

$f(\varphi(u,v),\psi(u,v))|J(u,v)|$ is integrable over *G* and

$$\iint_F f(x,y)\,dxdy=\iint_G f(\varphi(u,v),\psi(u,v))|J(u,v)|\,dudv$$

We will show why the Jacobian determinant of the transformation $T_{\varphi,\psi}$ actually appears in the preceding formula. This has something to do with the way the elementary partition areas change their form and surface area through the transformation.



$$C=[\varphi(u,v),\psi(u,v)]$$

$$D=[\varphi(u+c,v),\psi(u+c,v)]$$

$$B=\left[\varphi\left(u+c,v+d\right),\psi\left(u+c,v+d\right)\right]$$

$$A=\left[\varphi\left(u,v+d\right),\psi\left(u,v+d\right)\right]$$

$$C'=[u,v]$$

$$D'=[u+c,v]$$

$$B'=[u+c,v+d]$$

$$A'=[u,v+d]$$

As the partition area A'B'C'D' becomes infinitesimal, the area of the curvilinear quadrangle ABCD will tend to that of ABCD composed of line segments. Let us calculate | ABCD |.

The area of the quadrangle ABCD may be thought of as composed of the areas of the triangles ACD and DBA. Now the area of the parallelepiped given by the vectors $\overrightarrow{CA},\overrightarrow{CD}$ is twice that of the triangle ACD and the same holds for the triangle DBA and the parallelepiped given by the vectors $\overrightarrow{BA},\overrightarrow{BD}$

It is a well-known fact that the area of the parallelepiped given by the vectors $\overrightarrow{CA},\overrightarrow{CD}$ can be calculated as the absolute value of the determinant of a matrix whose lines are formed by the vectors $\overrightarrow{CA},\overrightarrow{CD}$. The same of course applies to the vectors $\overrightarrow{BA},\overrightarrow{BD}$ and thus we can write

$$|ABCD| = \left| \frac{\det\left(\overrightarrow{CA},\overrightarrow{CD}\right) + \det\left(\overrightarrow{BA},\overrightarrow{BD}\right)}{2} \right|$$

Let us now calculate the vectors $\overrightarrow{CA}, \overrightarrow{CD}, \overrightarrow{BA}, \overrightarrow{BD}$

$$\overrightarrow{CA} = \left(\varphi\left(u, v+d\right) - \varphi(u, v), \psi\left(u, v+d\right) - \psi(u, v)\right)$$

$$\overrightarrow{CD} = (\varphi(u+c, v) - \varphi(u, v), \psi(u+c, v) - \psi(u, v))$$

$$\overrightarrow{BA} = \left(\varphi\left(u, v+d\right) - \varphi\left(u+c, v+d\right), \psi\left(u, v+d\right) - \psi\left(u+c, v+d\right)\right)$$

$$\overrightarrow{BD} = \left(\varphi(u+c, v) - \varphi\left(u+c, v+d\right), \psi(u+c, v) - \psi\left(u+c, v+d\right)\right)$$

The mean value theorem tells us that there exist points I, J, K, L as shown in the preceding figure such that

$$\overrightarrow{CA} = d\left(\varphi_v'\left(K\right), \psi_v'\left(K\right)\right), \overrightarrow{CD} = c\left(\varphi_u'\left(L\right), \psi_u'\left(L\right)\right) \text{ and}$$

$$\overrightarrow{BA} = c\left(\varphi_u'\left(I\right), \psi_u'\left(I\right)\right), \overrightarrow{BD} = d\left(\varphi_v'\left(J\right), \psi_v'\left(J\right)\right)$$

Thus we have

$$\det\left(\overrightarrow{CA},\overrightarrow{CD}\right)=cd\begin{vmatrix}\varphi_v'(K) & \psi_v'(K)\\ \varphi_u'(L) & \psi_u'(L)\end{vmatrix}$$

$$\det\left(\overrightarrow{BA},\overrightarrow{BD}\right)=cd\begin{vmatrix}\varphi_u'(I) & \psi_u'(I)\\ \varphi_v'(J) & \psi_v'(J)\end{vmatrix}$$

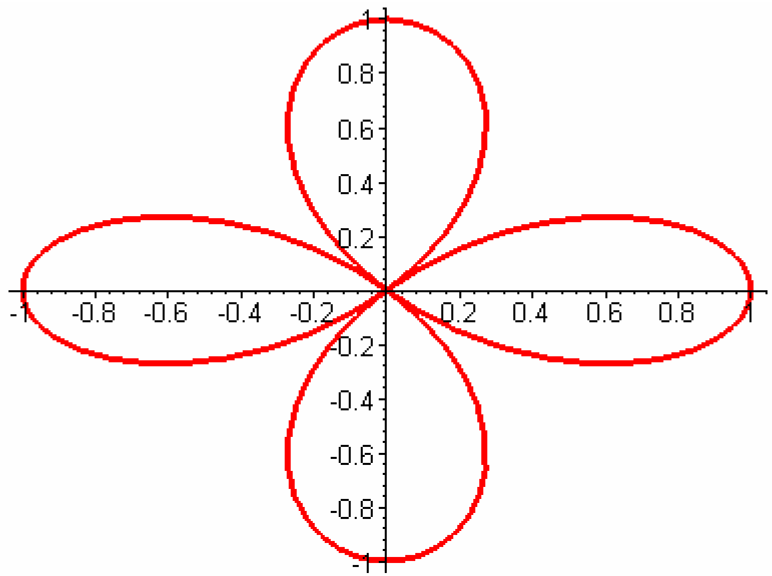
With the partition area becoming infinitesimal, the originally different points I,J,K,L will tend to one common point Q and so finally we can write

$$|ABCD|=\left|\frac{cd\begin{vmatrix}\varphi_v'(Q) & \psi_v'(Q)\\ \varphi_u'(Q) & \psi_u'(Q)\end{vmatrix}+cd\begin{vmatrix}\varphi_u'(Q) & \psi_u'(Q)\\ \varphi_v'(Q) & \psi_v'(Q)\end{vmatrix}}{2}\right|=cd\left\|\begin{matrix}\varphi_u'(Q) & \psi_u'(Q)\\ \varphi_v'(Q) & \psi_v'(Q)\end{matrix}\right\|=cd\,|J(Q)|$$

Example 1

Find the the surface area of the four petal rose shown in the figure below given by the equation

$$\left(x^2+y^2\right)^{3/2}=x^2-y^2 \qquad \textbf{(1)}$$



Since, with Cartesian coordinates, it would be very difficult to find the explicit expression of *y* as a function of *x*, we will try using the polar coordinates.

Indeed, substituting in (1) $\rho\cos\varphi$ for *x* and $\rho\sin\varphi$ for *y* yields: $\rho^3=\rho^2\cos 2\varphi$ or $\rho=\cos 2\varphi$ Since the figure has four axes of symmetry, we only need to calculate the area of F_1, which is half of the eastern petal and multiply the result by 8. In this case the area *G* consists of the points $[\rho,\varphi]\in[0,1]\times\left[\pi/4\right]$. The Jacobian determinant for the polar coordinates is ρ and so we can write $\left|F_1\right|=8\iint\limits_G \rho\, d\rho$ The double integral can be transformed into an iterated one as follows

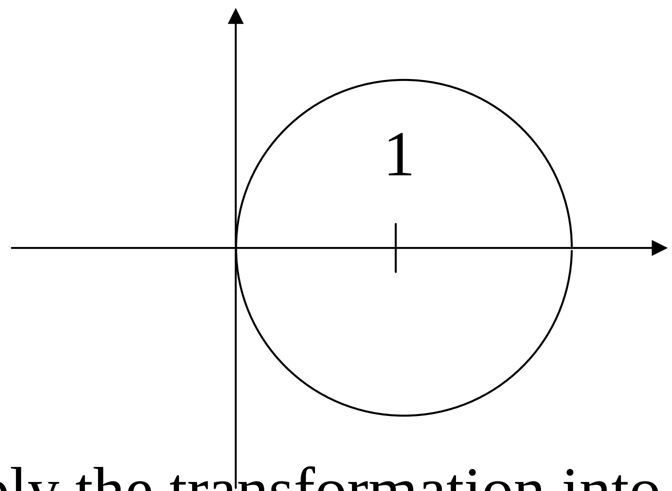
$$\iint\limits_G \rho\, d\rho\, d\varphi=\int_0^{\pi/4} d\varphi \int_0^{\cos 2\varphi} \rho\, d\rho=\int_0^{\pi/4}\left[\frac{\rho^2}{2}\right]_0^{\cos 2\varphi} d\rho=\frac{1}{2}\int_0^{\pi/4}\cos^2 2\varphi\, d\varphi$$

Using the formula $\cos^2 2x=\dfrac{1+\cos 4x}{2}$ we finally get $\left|F_1\right|=\pi$

Example 2

Calculate the double integral $\iint\limits_{M} x\,dx\,dy$ where M is the inside of the circle $(x-1)^2+y^2=1$.

Solution 1



Let us apply the transformation into polar coordinates. By substituting into the equation of the circle, we obtain

$(\rho\cos\varphi+1)^2+(\rho\sin\varphi)=1$ and thus $\rho=2\cos\varphi$

$$\iint_M x\,dx\,dy = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2\cos\varphi} \rho^2 \cos\varphi\, d\rho = \frac{8}{3}\int_{-\pi/2}^{\pi/2} \cos^4\varphi\, d\varphi =$$

$$= \int_{-\pi/2}^{\pi/2} \left(1 + \frac{4}{3}\cos 2\varphi + \frac{1}{3}\cos 4\varphi\right) d\varphi =$$

$$= \left[\varphi\right]_{-\pi/2}^{\pi/2} + \frac{2}{3}\left[\sin 2\varphi\right]_{-\pi/2}^{\pi/2} + \frac{1}{12}\left[\sin 4\varphi\right]_{-\pi/2}^{\pi/2} = \pi + 0 + 0 = \pi$$

Solution 2

Let us apply the slightly modified polar coordinates

$x = \rho\cos\varphi + 1,\ \ y = \rho\sin\varphi$ where the Jacobian determinant is again ρ and the area G is the rectangle $\left[0,1\right]\times\left[0,2\pi\right]$.

$$\iint_M x\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^1 \rho^2\cos\varphi + \rho\,d\rho = \int_0^{2\pi} \left[\frac{\rho^3\cos\varphi}{3}\right]_0^1 + \left[\frac{\rho^2}{2}\right]_0^1 d\varphi =$$

$$\int_0^{2\pi} \frac{1}{3}\cos\varphi + \frac{1}{2}\,d\varphi = 0 + \pi = \pi$$

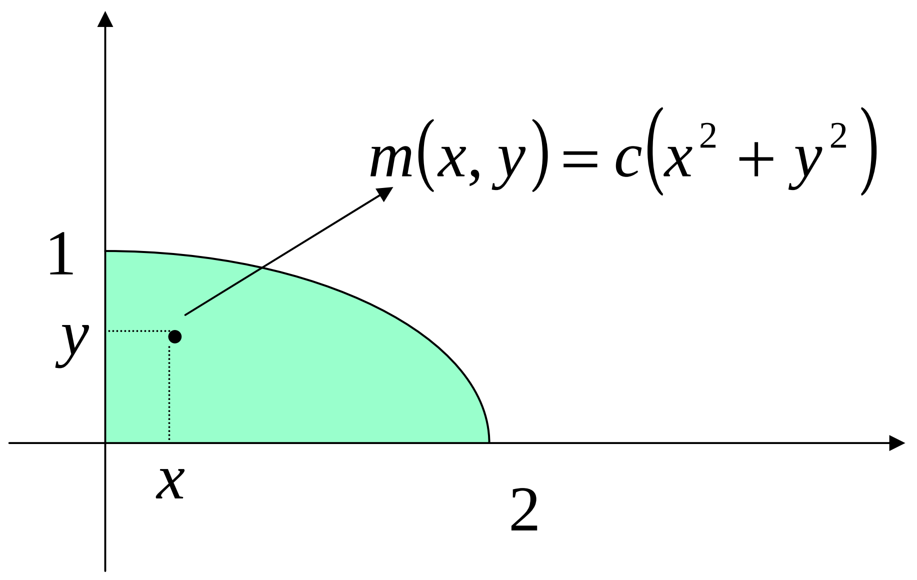
This example demonstrates that sometimes a subtle change in the system of coordinates chosen may be quite of advantage.

Example 3

Find the centre of gravity of a thin membrane whose shape is given by the following inequalities:

$$x \geq 0,\ y \geq 0,\ x^2 + 4y^2 - 4 \leq 0$$

further we know that the specific mass of the membrane is directly proportionate to the distance from the origin with a constant $c > 0$.



For coordinates [t_x, t_y] of the center of gravity of a planar area M with $m(x,y)$ as the specific mass, we have

$$t_x = \frac{\iint\limits_M x\, m(x,y)\, dx\, dy}{\iint\limits_M m(x,y)\, dx\, dy}, \quad t_y = \frac{\iint\limits_M y\, m(x,y)\, dx\, dy}{\iint\limits_M m(x,y)\, dx\, dy}$$

To calculate these double integrals we will use weighted polar coordinates $x = 2\rho\cos\varphi, \quad y = \rho\sin\varphi$ with the Jacobian determinant $J(\rho,\varphi) = 2\rho.$ Then the area G will be the rectangle $[0,1]\times[0,\pi/2]$

$$\iint\limits_m c\left(x^2 + y^2\right) dx\, dy = c\int_0^{\pi/2}\left(8\cos^2\varphi + 2\sin^2\varphi\right) d\varphi \int_0^1 \rho^3 d\rho =$$

$$= c\int_0^{\pi/2}(5+3\cos 2\varphi)\,d\varphi\int_0^1 \rho^3 d\rho = \frac{5c\pi}{2}+\frac{3c}{2}\left[\sin 2\varphi\right]_0^{\pi/2} = \frac{5c\pi}{2}$$

$$\iint_M c\left(x^2+y^2\right)x\,dx\,dy = c\int_0^{\pi/2}\left(16\cos^3\varphi+4\sin^2\varphi\right)d\varphi\int_0^1 \rho^4\, d\rho =$$

$$= \frac{c}{5}\int_0^1 16\left(1-t^2\right)+4t^2 = \frac{16c}{5}\left[t-\frac{t^3}{3}\right]_0^1 + \frac{4c}{5}\left[\frac{t^3}{3}\right]_0^1 = \frac{36c}{15}$$

In the last integral, we used the $\sin\varphi = t$ transformation.

Now we can calculate

$$t_x = \frac{\frac{36c}{15}}{\frac{5c\pi}{12}} = \frac{144}{25\pi} \approx 1{,}83346$$

$$\iint_M c\left(x^2+y^2\right)y\,dx\,dy = c\int_0^{\pi/2}\left(8\cos^2\varphi\sin\varphi + 2\sin^3\varphi\right)d\varphi\int_0^1 \rho^4\,d\rho =$$

$$= \frac{c}{5}\int_0^1 8t^2 + 2\left(1-t^2\right) = \frac{8c}{5}\left[\frac{t^3}{3}\right]_0^1 + \frac{2c}{5}\left[1-\frac{t^3}{3}\right]_0^1 = \frac{12c}{15}$$

And so

$$t_y = \frac{\frac{12c}{15}}{\frac{5c\pi}{12}} = \frac{144}{75\pi} \approx 0.61115$$