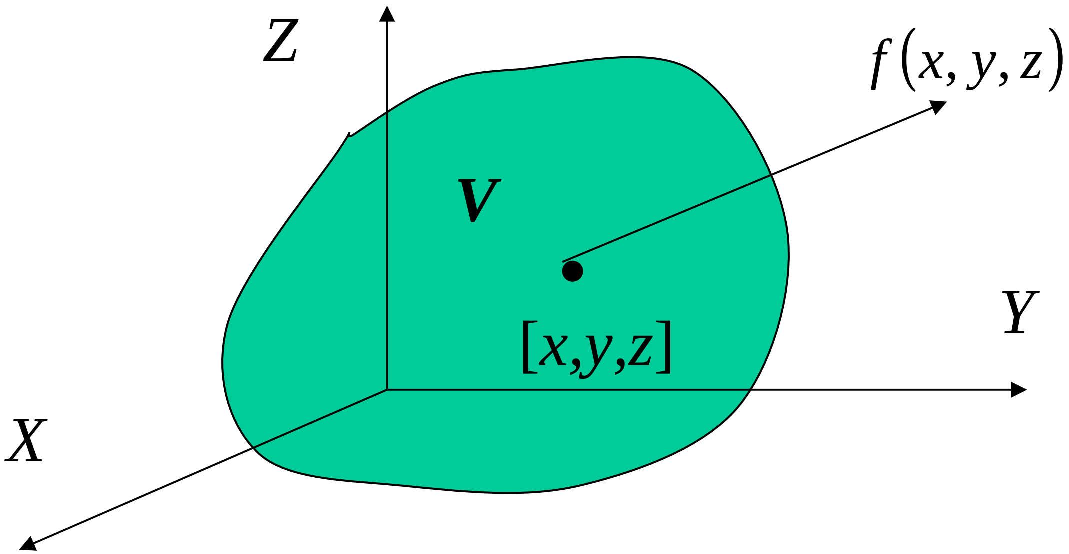
TRIPPLE INTEGRAL



For a bounded closed 3-D area V and a function $f(x, y, z)$ continuous on V, we define the triple integral of $f(x, y, z)$ with respect to V, in symbols

$$\iiint_V f(x, y, z)\, dx\, dy\, dz$$

in much the same manner as the double integral.

- A finite partition $\boldsymbol{P}$ of the 3-D area V is defined with partition subareas $P_1, P_2, \ldots, P_n$. By $|\,\boldsymbol{P}\,|$ we will denote the volume of $\boldsymbol{P}$, and by d($\boldsymbol{P}$) the smallest diameter of its subareas.

- For each subarea P_i we define m_i and M_i as the glb and lub of $f(x,y,z)$ on P_i

- For a partition $\boldsymbol{P}$, we define the lower and upper integral sums

$$I_L(\boldsymbol{P}) = \sum_{P_i \in \boldsymbol{P}} m_i |P_i|, \quad I_U(\boldsymbol{P}) = \sum_{P_i \in \boldsymbol{P}} M_i |P_i|,$$

- If $\lim\limits_{d(\boldsymbol{P})\to 0} I_L = \lim\limits_{d(\boldsymbol{P})\to 0} I_U$ we say that $f(x,y,z)$ is integrable over V

and put $$\lim_{d(\boldsymbol{P})\to 0} I_L = \lim_{d(\boldsymbol{P})\to 0} I_U = \iiint_V f(x,y,z)\,dx\,dy\,dz$$

PROPERTIES OF TRIPPLE INTEGRAL

The triple integral has additive properties similar to those of double integral.

Also the mean value theorem can be easily rephrased for triple integral:

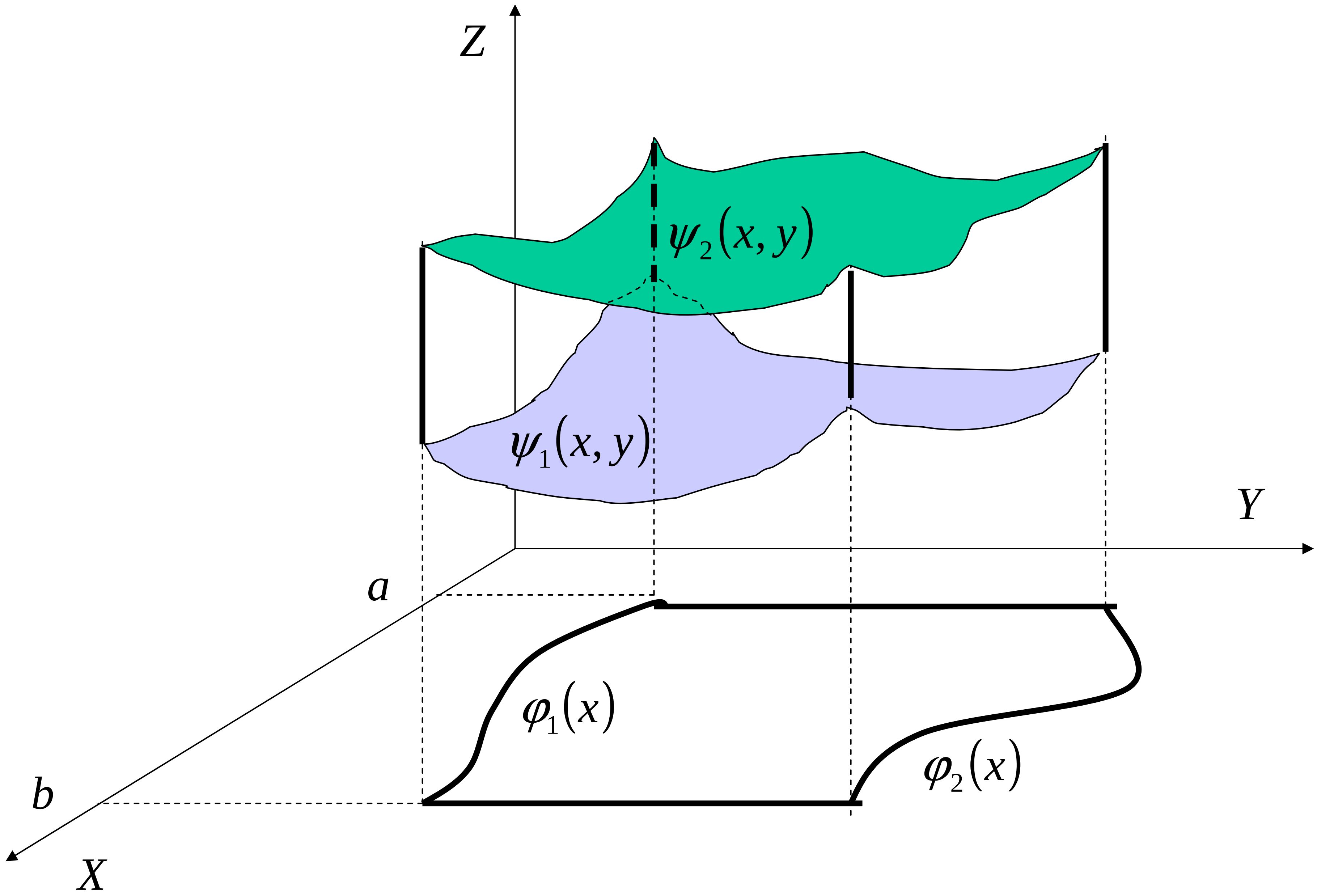
Let f(x,y,z) be integrable over V with $a \le f(x,y,z) \le b,$ then

$$\Rightarrow \quad a|V| \le \iiint\limits_V f(x,y,z)\,dx\,dy\,dz \le b|V|$$

$\Rightarrow$ there exists a $c,$ $a \le c \le b,$ such that

$$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz = c|V|$$

FUBINI'S THEOREM FOR TRIPPLE INTEGRAL



Let V be a 3-D area defined by the following inequalities

$$\psi_1(x,y) \le z \le \psi_2(x,y),\ \varphi_1(x) \le y \le \varphi_2(x),\ a \le x \le b$$

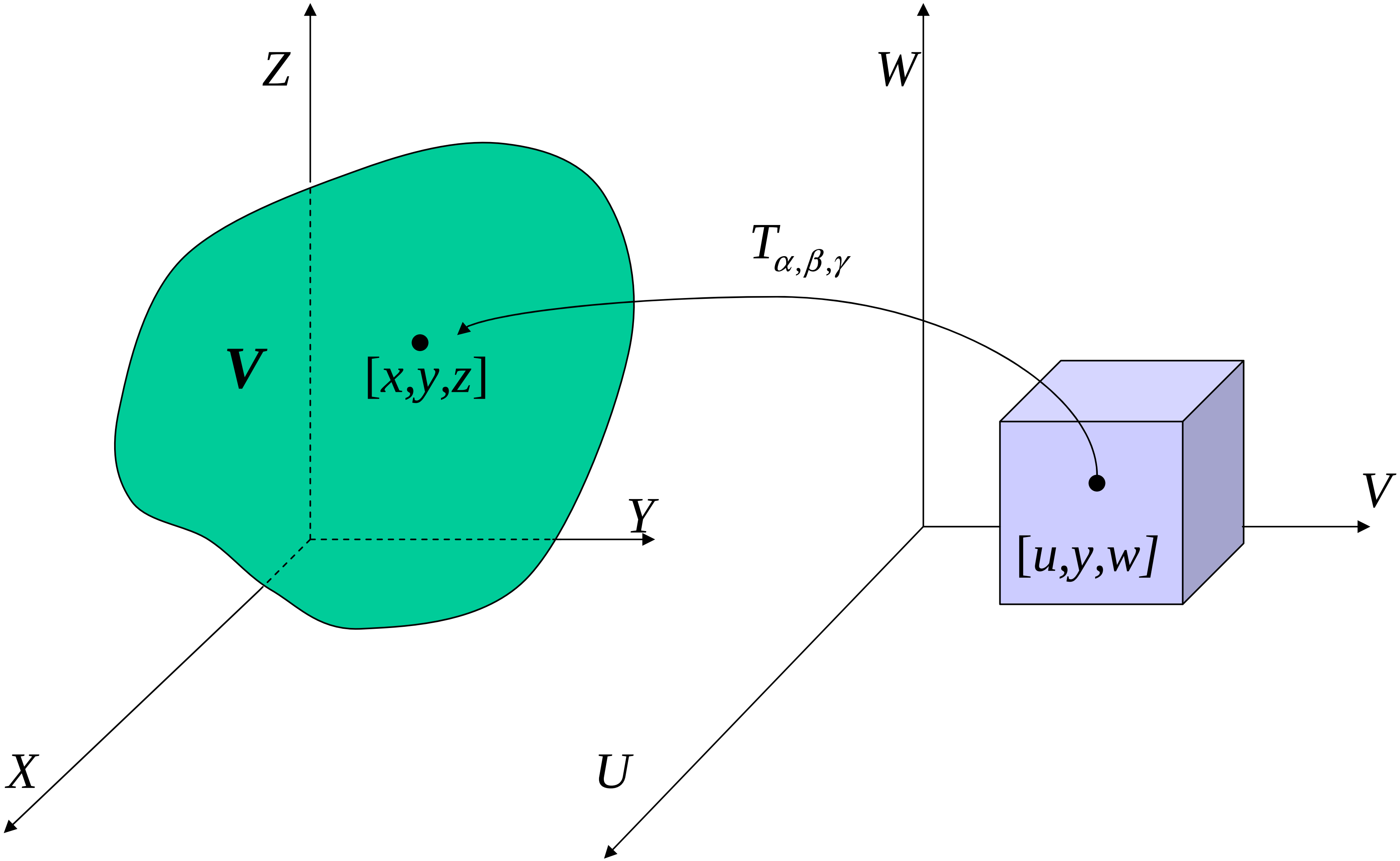
and let f(x,y,z) be integrable with respect to V.

If $\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz$ exists for every $a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)$

and $\int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz$ exists for every $a \le x \le b$ then

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz$$

TRANSFORMATION FOR TRIPLE INTEGRAL



$$T_{\alpha,\beta,\gamma} : \boldsymbol{R}^3 \to \boldsymbol{R}^3$$

$$T_{\alpha,\beta,\gamma} : [\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] = [x, y, z]$$

$$x = \alpha(u, v, w)$$

$$y = \beta(u, v, w)$$

$$z = \gamma(u, v, w)$$

The Jacobian determinant $J(u,v,w)$ for the transformation $T_{\alpha,\beta,\gamma}$

$$J(u,v,w) = \begin{vmatrix} \alpha_u'(u,v,w) & \beta_u'(u,v,w) & \gamma_u'(u,v,w) \\ \alpha_v'(u,v,w) & \beta_v'(u,v,w) & \gamma_v'(u,v,w) \\ \alpha_w'(u,v,w) & \beta_w'(u,v,w) & \gamma_w'(u,v,w) \end{vmatrix}$$

If U, V are closed bounded 3-D areas $T_{\alpha,\beta,\gamma}(U) = V$ and $J(u,v,w) \neq 0$ in the interior of U, we say that $T_{\alpha,\beta,\gamma}$ is a regular transformation of U onto V.

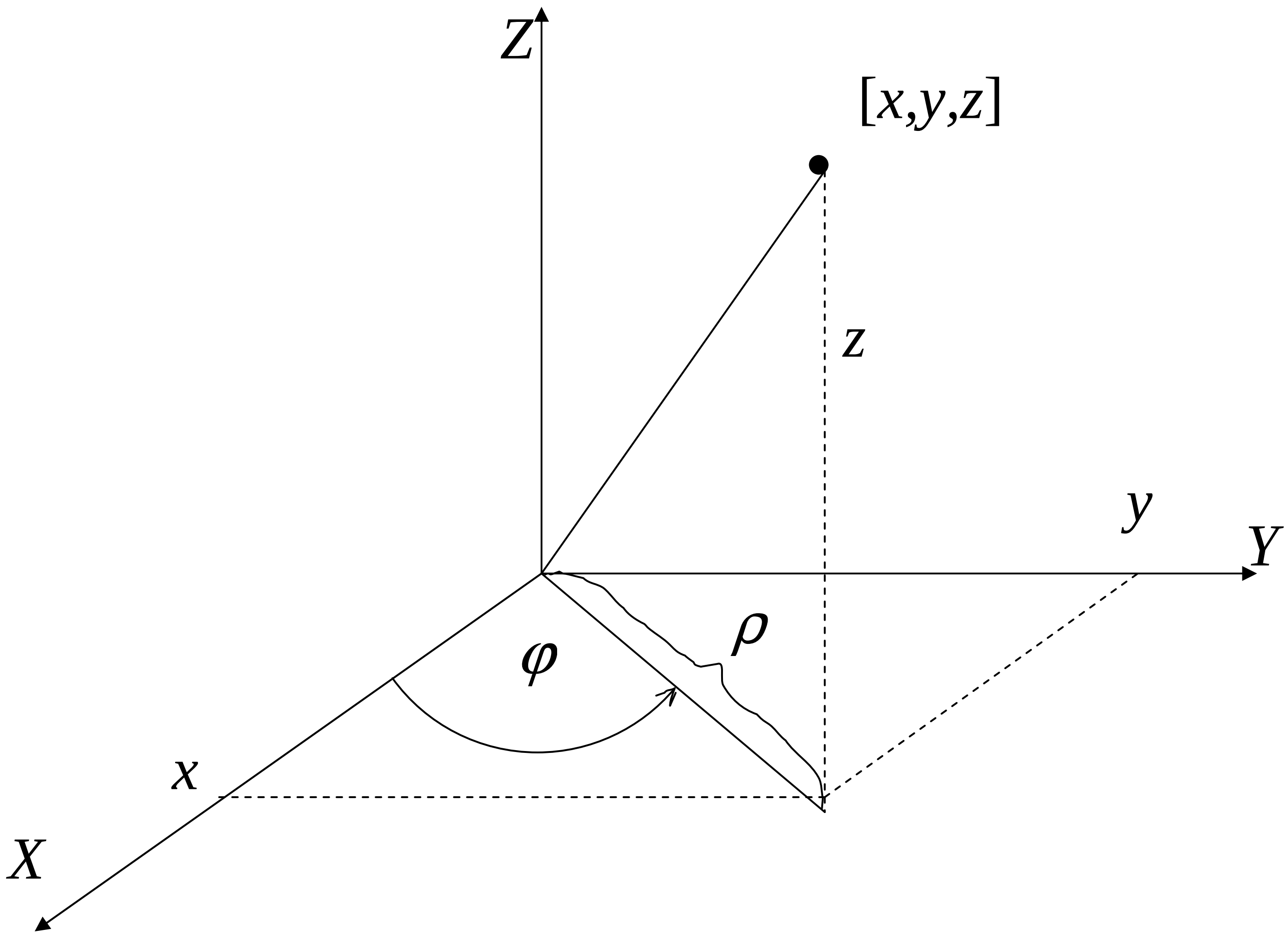
CHANGE OF VARIABLES IN TRIPLE INTEGRAL

Let f(x,y,z) be integrable over a closed bounded area V and let

$T_{\alpha,\beta,\gamma}$ be a regular transformation of *U* onto *V*. Then

$$\iiint\limits_V f(x, y, z)\, dx\, dy\, dz = \iiint\limits_U f(\alpha(u, v, w), \beta(u, y, w), \gamma(u, v, w)) \left| J(u, v, w) \right| du\, dv\, dw$$

CYLINDRICAL CO-ORDINATES IN R^3



TRANSFORMATION USING CYLINDRICAL CO-ORDINATES

$$T_{\alpha,\beta,\gamma}\left[\rho,\varphi,z\right]=\left[x,y,z\right]$$

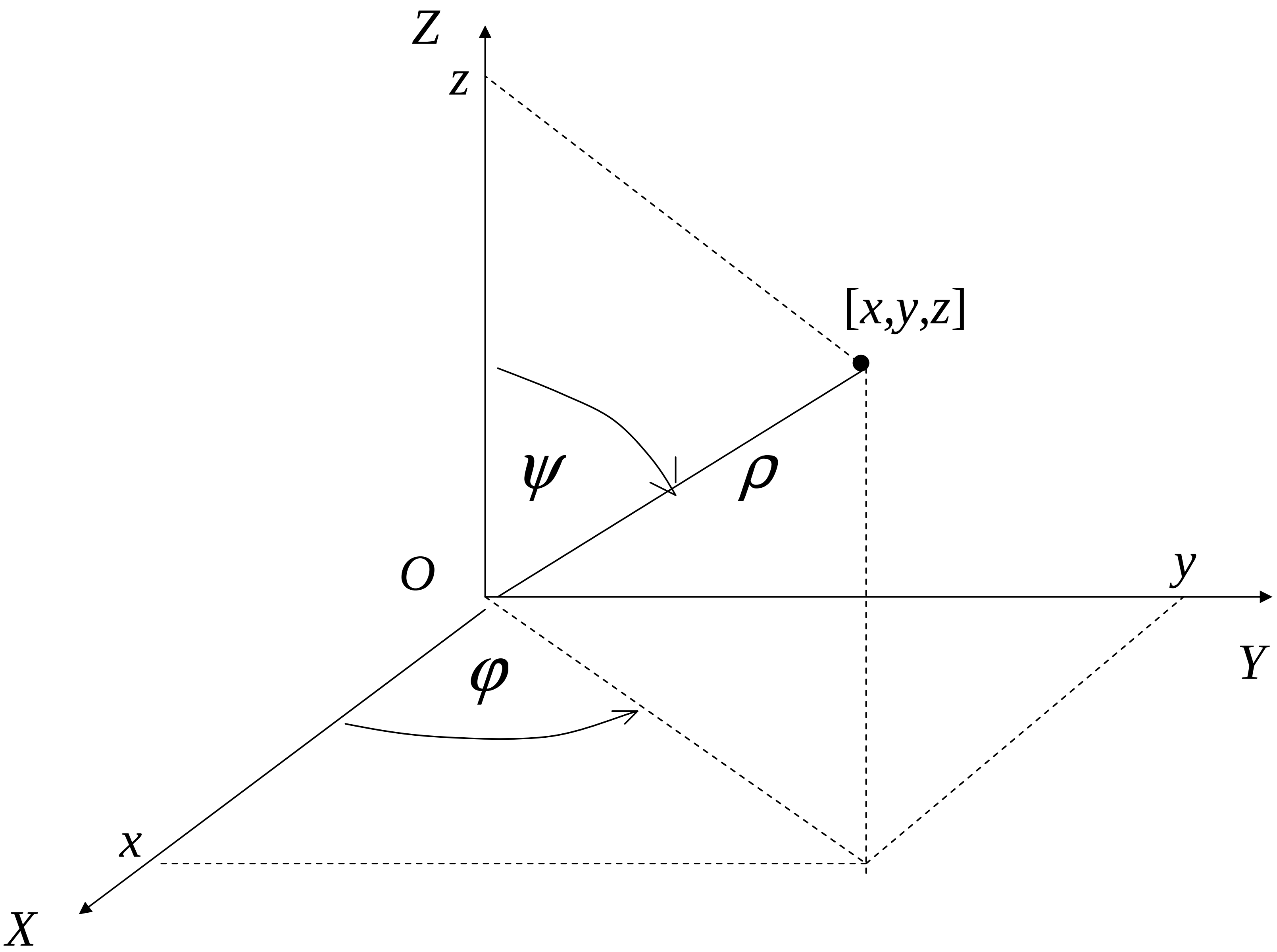
$$x=\rho\cos\varphi$$

$$y=\rho\sin\varphi$$

$$z=z$$

$$J\left(\rho,\varphi,z\right)=\begin{vmatrix}\cos\varphi & \sin\varphi & 0\\ -\rho\sin\varphi & \rho\cos\varphi & 0\\ 0 & 0 & 1\end{vmatrix}=\rho$$

SPHERICAL CO-ORDINATES



TRANSFORMATION USING SPHERICAL CO-ORDINATES

$$T_{\alpha,\beta,\gamma}\left[\varphi,\psi,\rho\right]=\left[x,y,z\right]$$

$$x=\rho\cos\varphi\sin\psi$$

$$y=\rho\sin\varphi\sin\psi$$

$$x=\rho\cos\psi$$

$$J\left(\rho,\varphi,\psi\right)=\begin{vmatrix} \cos\varphi\sin\psi & \sin\varphi\sin\psi & \cos\psi \\ -\rho\sin\varphi\sin\psi & \rho\cos\varphi\sin\psi & 0 \\ \rho\cos\varphi\cos\psi & \rho\sin\varphi\cos\psi & -\rho\sin\psi \end{vmatrix}=-\rho^2\sin\psi$$

WEIGHTED SPHERICAL CO-ORDINATES

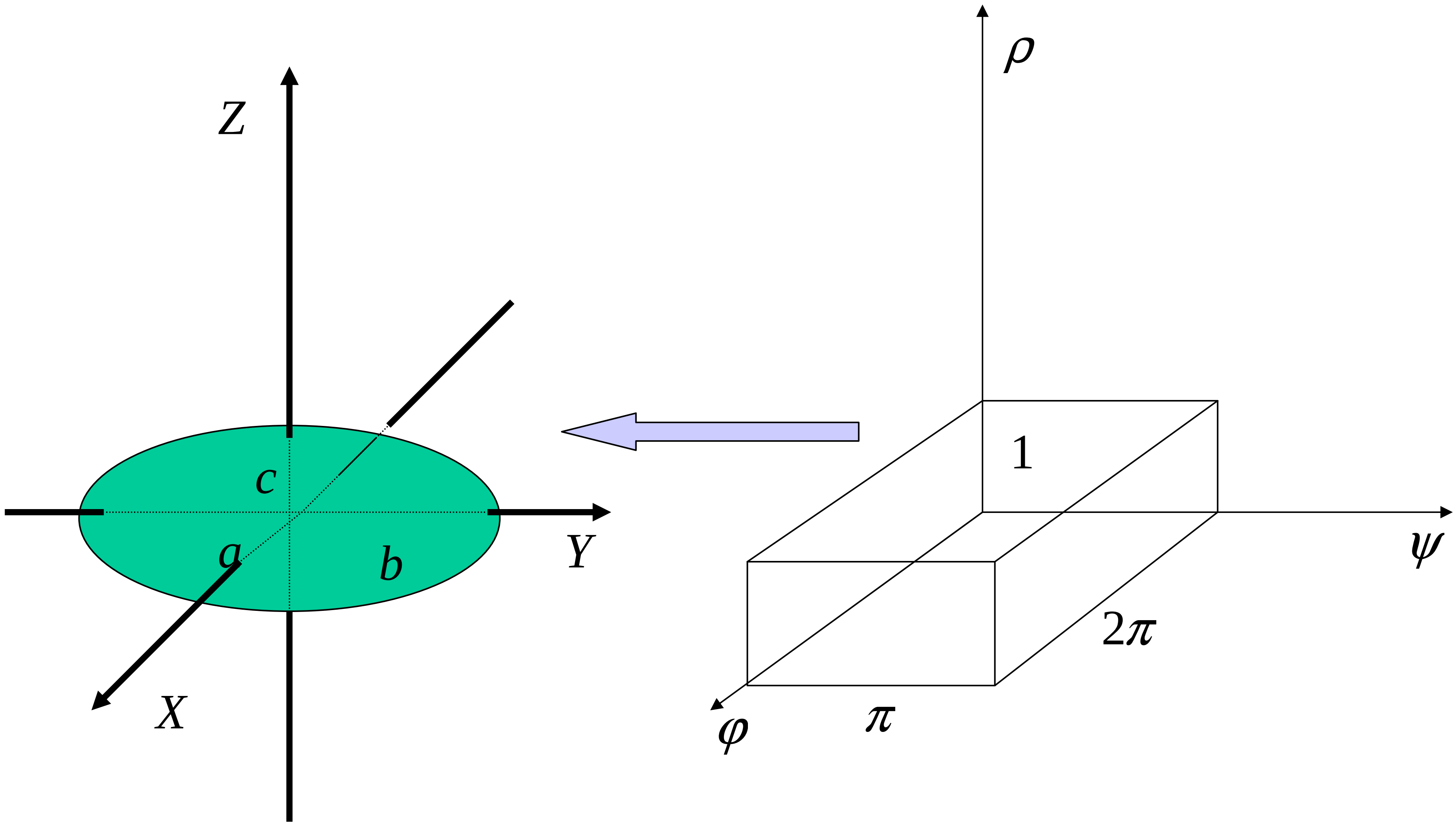
$$T_{\alpha,\beta,\gamma}[\varphi,\psi,\rho]=[x,y,z]$$

$$x=a\rho\cos\varphi\sin\psi \qquad a>0,\ b>0,\ c>0$$

$$y=b\rho\sin\varphi\sin\psi$$

$$x=c\rho\cos\psi$$

$$J(\rho,\varphi,\psi)=\begin{vmatrix} a\cos\varphi\sin\psi & b\sin\varphi\sin\psi & c\cos\psi \\ -a\rho\sin\varphi\sin\psi & b\rho\cos\varphi\sin\psi & 0 \\ a\rho\cos\varphi\cos\psi & b\rho\sin\varphi\cos\psi & -c\rho\sin\psi \end{vmatrix}=-abc\rho^2\sin\psi$$


Z
c
a
b
Y
X
ρ
1
ψ
2π
π
φ

EXAMPLE

Find the mass of an ellipsoid with half-axes a,b,c where the specific mass at a point $[x,y,z]$ is given as $|x|+|y|+|z|$.

Solution

We will consider an ellipsoid E given by the inequality

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\leq 1$$

Clearly, the specific mass is symmetric with respect to the planes xy, xz, and yz and so we may only calculate the mass of an eighth of the ellipsoid, say, for $x,y,z\leq 0.$

If we use a transformation T using the weighted spherical co-ordinates

$$x = a\rho\cos\varphi\sin\psi$$

$$y = b\rho\sin\varphi\sin\psi$$

$$x = c\rho\cos\psi$$

we can think of E as the image of T where the domain of T is a rectangular parallelepiped R $[0,1]\times\left[\pi/2\right]\times\left[\pi/2\right]$

The mass M then equals

$$M = 8abc\int_0^1 d\rho\int_0^{\pi/2} d\varphi\int_0^{\pi/2}\left(\rho\cos\varphi\sin\psi + \rho\sin\varphi\sin\psi + \rho\cos\psi\right)\rho^2\sin\psi\, d\psi$$

which can be written as

$$M = 8abc\int_0^1 \rho^3 d\rho\left(CS_2 + SS_2 + CS\right)$$

where

$$C = \int_0^{\pi/2} \cos t\, dt, \quad S = \int_0^{\pi/2} \sin t\, dt, \quad S_2 = \int_0^{\pi/2} \sin^2 t\, dt$$

$$C = S = 1$$

By a simple calculation we can determine that

$C = S = 1$ and $S_2 = \pi/4$.

This yields

$$M = 8abc\left(\frac{1}{4}\left(\frac{\pi}{4} + \frac{\pi}{4} + 1\right)\right) = abc(\pi + 2)$$