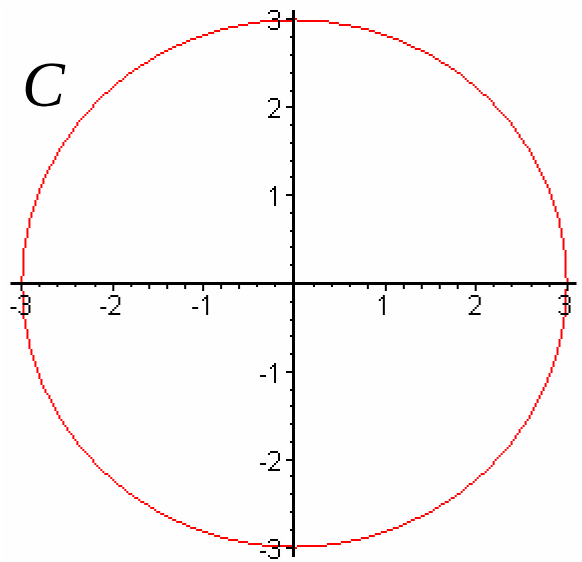
# PLANAR CURVES

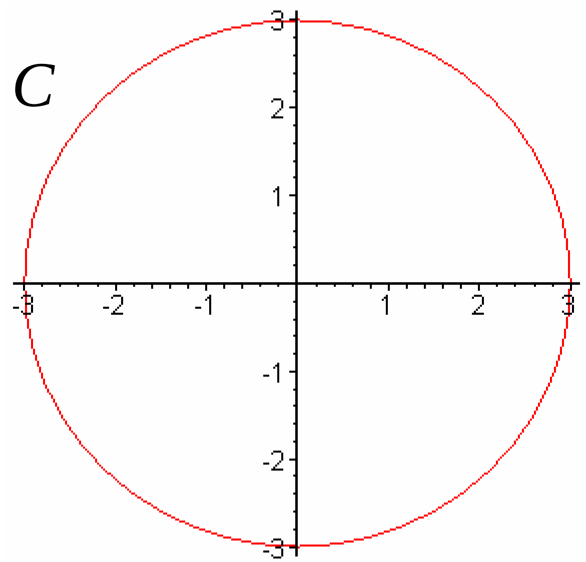
A circle of radius 3 can be expressed in a number of ways




Using parametric equations:

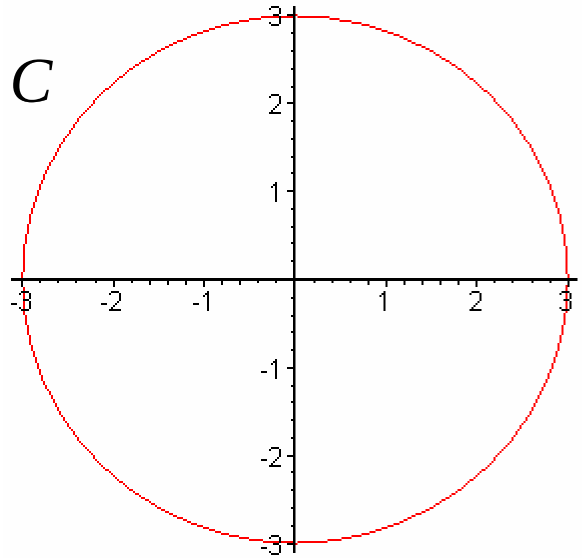
$$C = \{ [x, y];\ x = 3\cos t, \quad y = 3\sin t, \quad t \in [-\pi, \pi) \}$$

Using an implicit function:



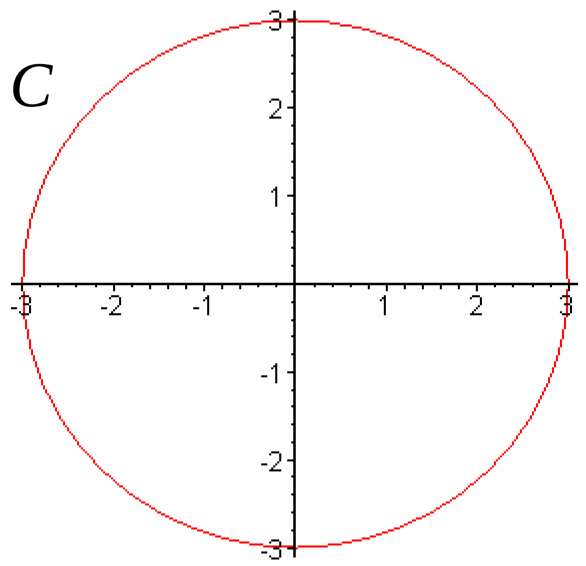

$$C = \left\{ [x, y];\ x^2 + y^2 = 9 \right\}$$

Using an explicit function:




$$C = \left\{ [x, y];\, y = \sqrt{9 - x^2} \right\} \cup \left\{ [x, y];\, y = -\sqrt{9 - x^2} \right\}$$

Using polar co-ordinates:




$$C = \{ [\rho, \varphi];\ \rho = 3 \}$$

# SMOOTH REGULAR CURVE

Let a curve $C$ be given by the parametric equations

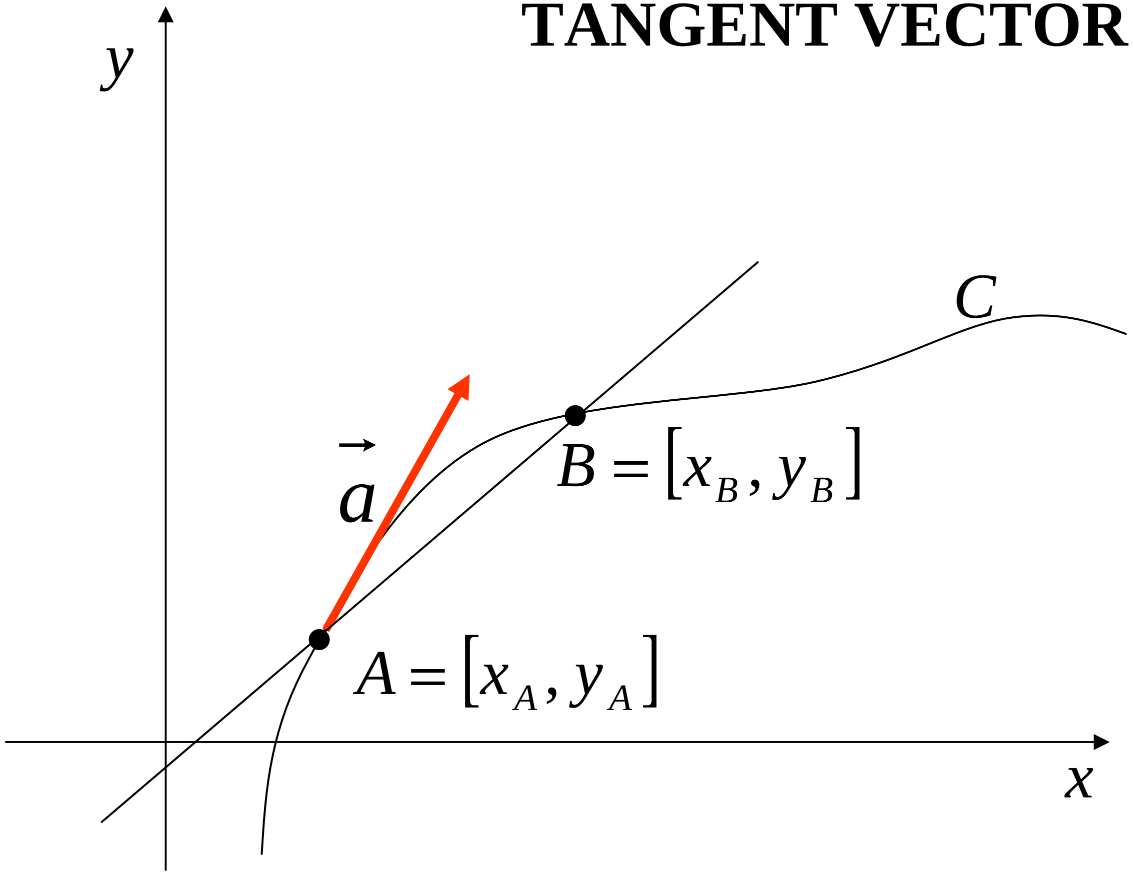
$$C = \{[x, y];\ x = \varphi(t), \quad y = \psi(t), \quad t \in [a, b]\}$$

We say that $C$ is a ***smooth regular curve*** if

- $\varphi'(t), \psi'(t)$ exist and are continuous on $[a,b]$
- $\varphi'(t)\psi'(t) \neq 0$ on $[a,b]$,
- for each point $[x,y]$ of $C$, there is a unique $t \in [a,b]$ such that $x = \varphi(t), \quad y = \psi(t)$

Any point of $C$ for which any of the above conditions is not satisfied is called ***singular*** with a possible exception of the points corresponding to the values $a$ and $b$.

# TANGENT VECTOR




$$\overrightarrow{AB} \to \vec{a} \quad \text{and} \quad \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} \to \frac{\vec{a}}{\left|\vec{a}\right|} \text{ as } A \text{ tends to } B$$

Let $C$ be a smooth curve expressed parametrically

$$C = \{\, [x, y];\ x = \varphi(t), \quad y = \psi(t), \quad t \in I \,\}$$ with

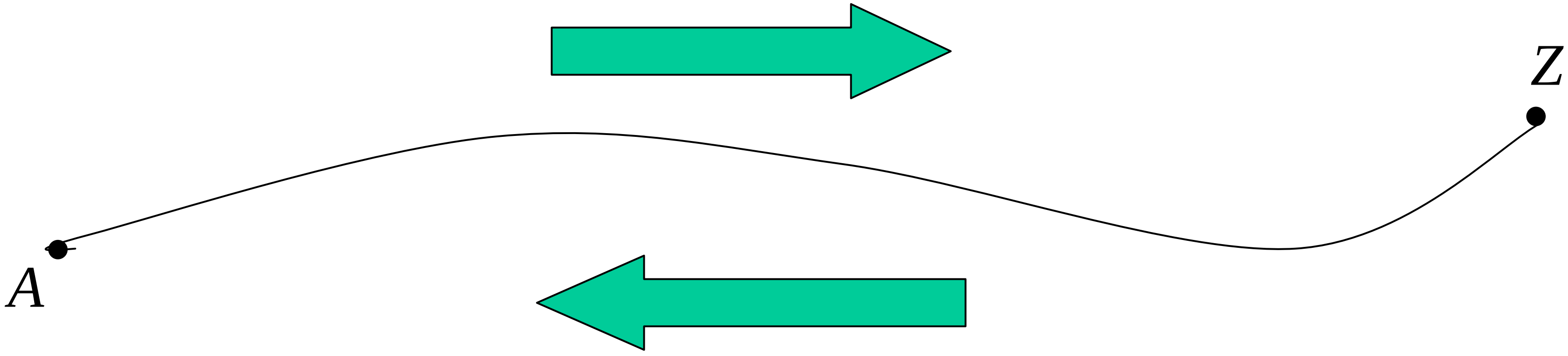
$$A = [\varphi(t_A), \psi(t_A)],\ B = [\varphi(t_B), \psi(t_B)].$$ We have

$$\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{\left(\varphi(t_A) - \varphi(t_B), \psi(t_A) - \psi(t_B)\right)}{\sqrt{\left(\varphi(t_A) - \varphi(t_B)\right)^2 + \left(\psi(t_A) - \psi(t_B)\right)^2}}$$

and, as $B$ tends to $A$, we get, using the mean value theorem:

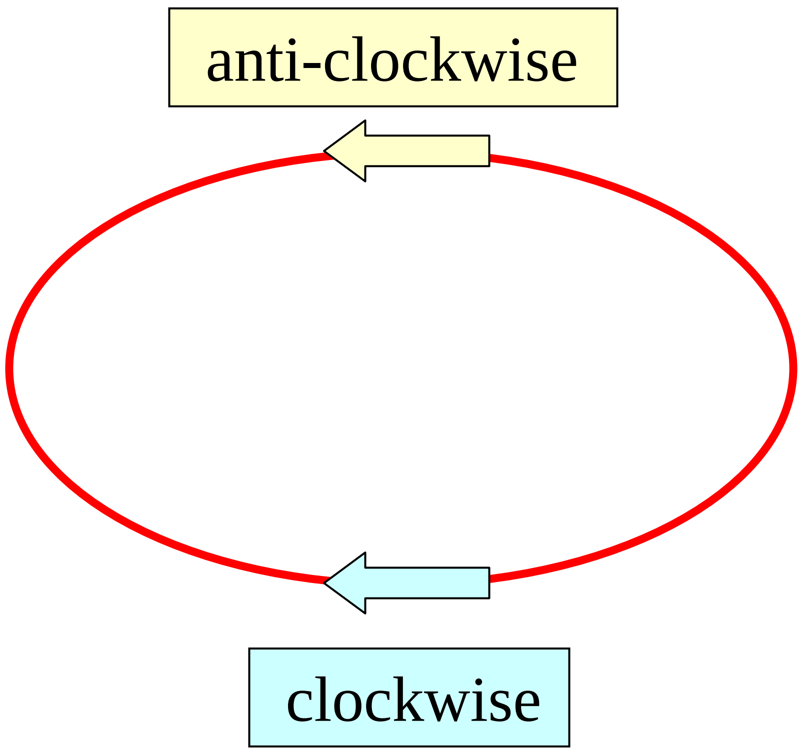
$$\frac{\vec{a}}{\left|\vec{a}\right|} = \frac{\left(\varphi'(t_A), \psi'(t_A)\right)}{\sqrt{\left(\varphi'(t_A)\right)^2 + \left(\psi'(t_A)\right)^2}}$$

# DIRECTING AN OPEN CURVE

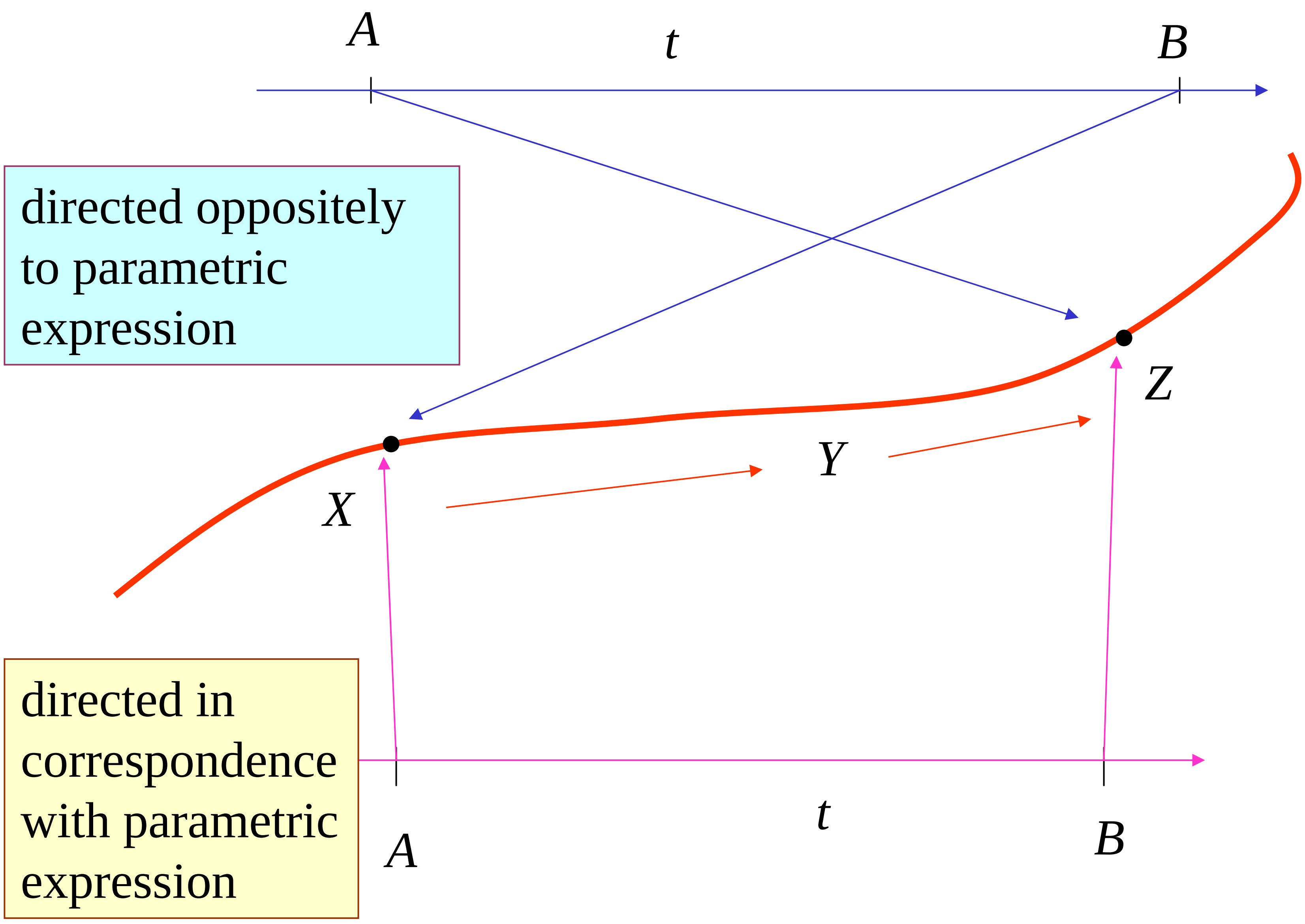



If a curve is open, directing it means saying which endpoint comes as first and which as second. There are clearly two ways or directing a curve.

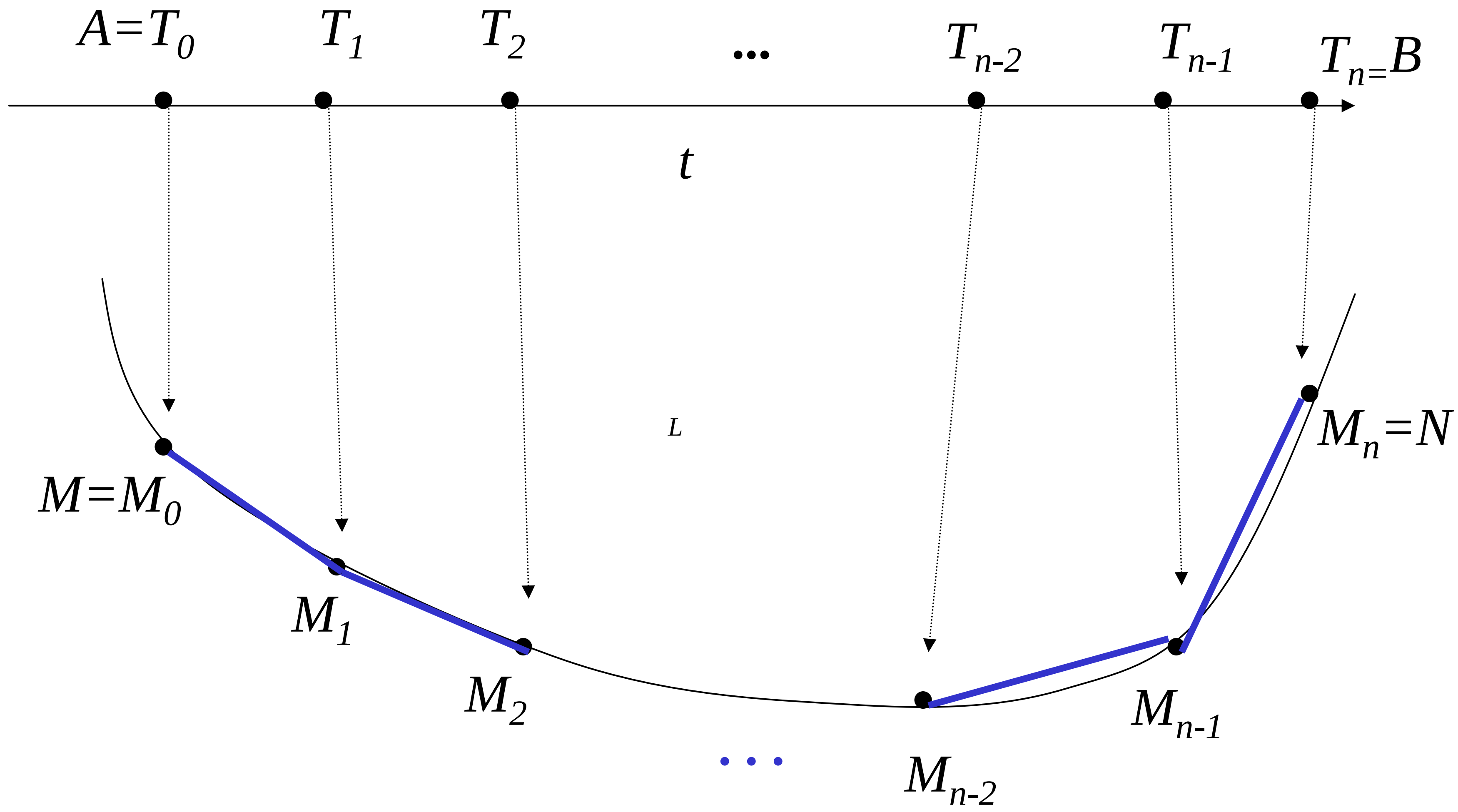
# DIRECTING A CLOSED CURVE



# DIRECTION AND PARAMETRIC EXPRESSION



# RECTIFIABLE CURVES




$$L_{T_0 \ldots T_n} = \overline{M_0 M_1} + \overline{M_1 M_2} + \cdots + \overline{M_{n-2} M_{n-1}} + \overline{M_{n-1} M_n}$$

$L_{T_0 \ldots T_n}$ is the length of the polygon inscribed into curve *MN*

with respect to partition $T_0$, $T_1$, $T_2$, ... $T_{n-2}$, $T_{n-1}$, $T_n$ of *AB*.

Let us now consider the set $\mathcal{L}$ of such lengths taken over the set of all the partitions of AB. If $\mathcal{L}$ is bounded from above, then we say that the curve *MN* is ***rectifiable.***

If *AB* is rectifiable, the least upper bound of $\mathcal{L}$ exists and we define this least upper bound *L* as the length of *AB*.

$$L(AB)=\mathrm{LUB}(\mathcal{L})$$

# LENGTH OF A REGULAR CURVE

Let $C$ be a regular curve with a parametric expression

$$C = \{[x, y];\ x = \varphi(t), \quad y = \psi(t), \quad t \in [a, b]\}$$

Then it is rectifiable and the length of C can be calculated by the following formula:

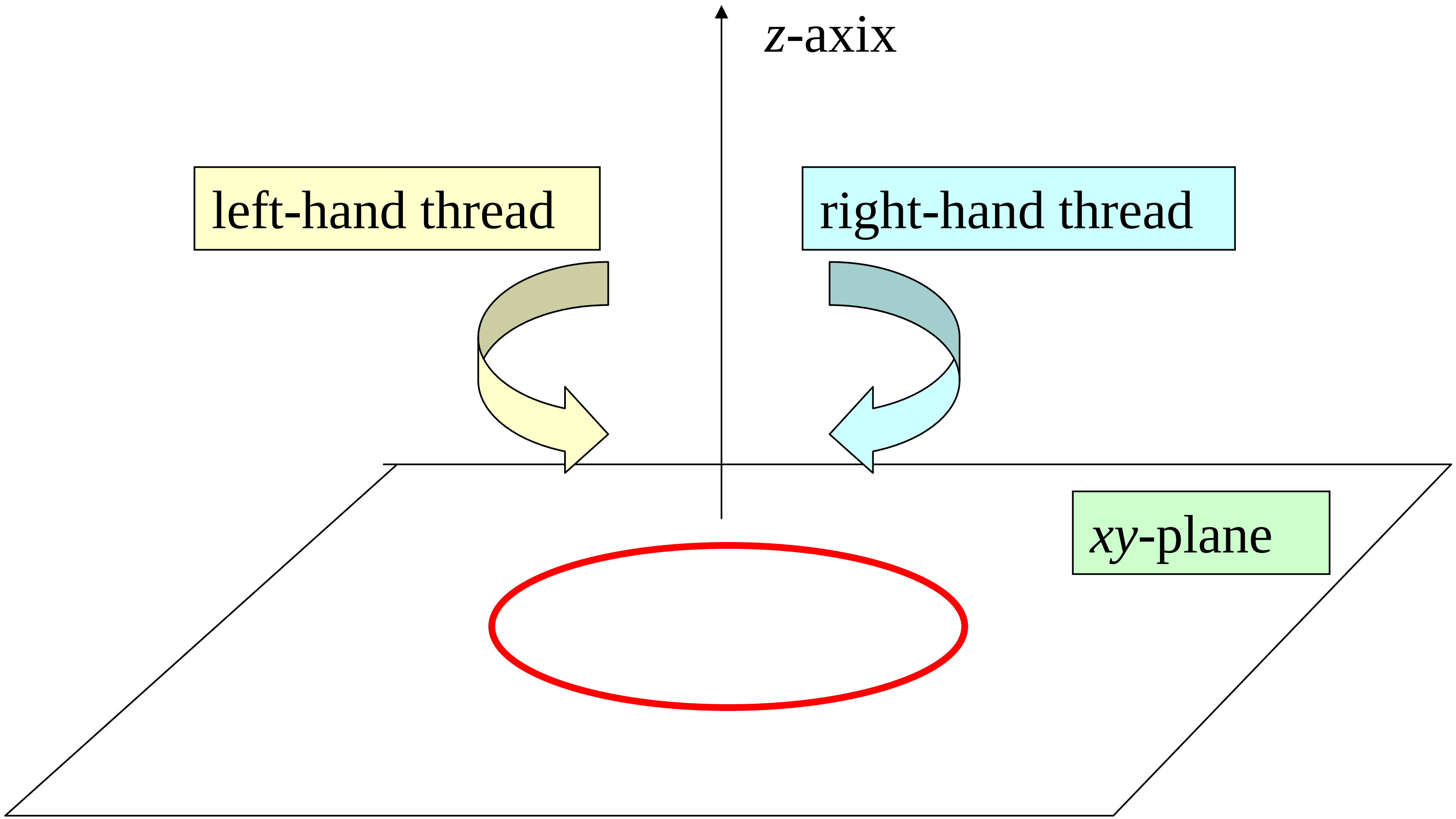
$$L(C) = \int_a^b \sqrt{(\varphi'(t))^2 + (\psi'(t))^2}\, dt$$

# 3-D CURVES

The notions of a tangent vector, the rectifiability, the length of a curve and the direction of anopen3-D curve are easily extended to a 3-D curve.

To direct a 3-D closed curve we use the notion of a left-hand and right-hand thread.

# DIRECTING A 3-D CLOSED CURVE

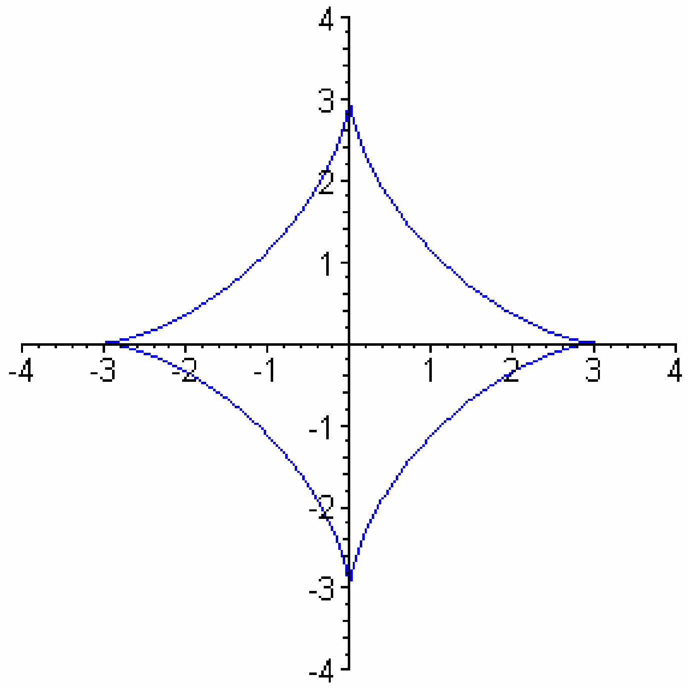


# ASTROID

$$x^{2/3} + y^{2/3} = a^{2/3} \quad \text{or} \quad \left(x^2 + y^2 - a^3\right)^3 + 27a^2x^2y^2 = 0$$

$$x = a\cos^3 t$$

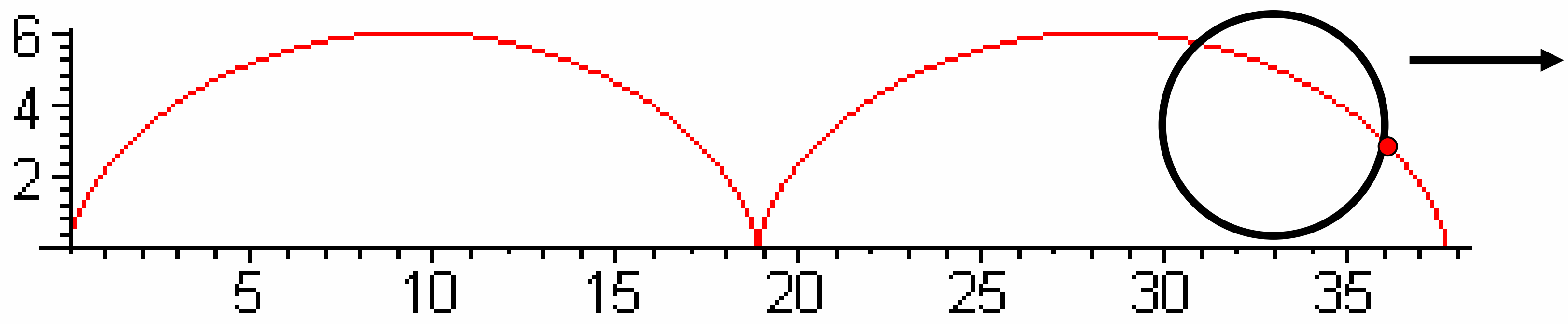
$$y = a\sin^3 t \quad t \in [0,\ 2\pi]$$



# CYCLOID

Cycloid is the curve generated by a point on the circumference of a circle that rolls along a straight line. If $r$ is the radius of the circle and $\varphi$ is the angular displacement of the circle, then the parametric equations of the curve are $x = r(\varphi - \sin\varphi),\ y = r(1 - \cos\varphi)$

Here $r$ =3 and φ runs from 0 to 4π, that is, the circle revolves twice



# CARDOID

$$\left(x^2 + y^2 - 2ax\right)^2 = 4a^2\left(x^2 + y^2\right)$$

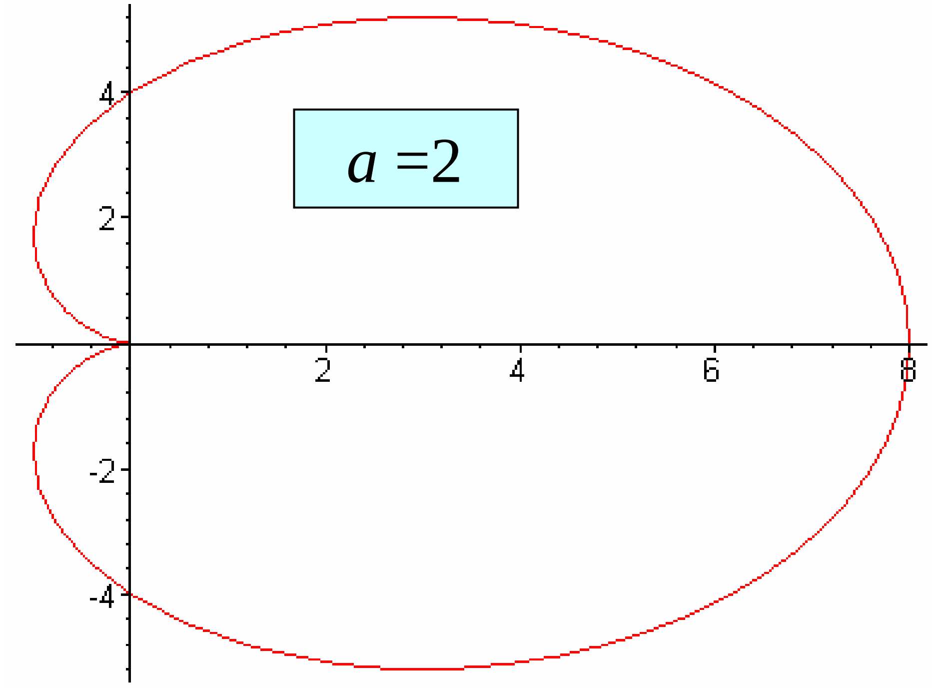
implicit function

$$\rho = 2a(1 + \cos\varphi)$$

polar co-ordinates

$$x = a(2\cos t - \cos 2t)$$
$$y = a(2\sin t - \sin 2t)$$

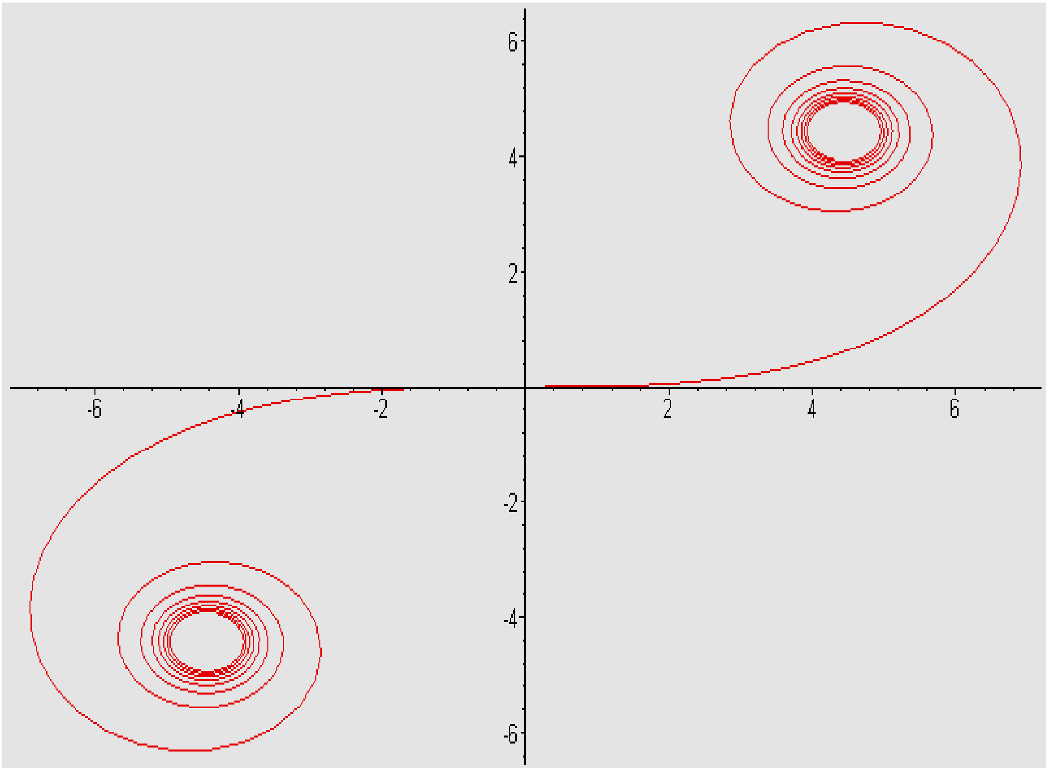
parametric equations



# CLOTHOIDE

Clothoide is a curve whose radius of curvature at a point *M* is indirectly proportional to the length s between M and a fixed point *O*.

$$x = \int_0^s \cos\frac{t^2}{2a^2}dt, \quad y = \int_0^s \sin\frac{t^2}{2a^2}dt$$



# FOLIUM OF DESCARTES

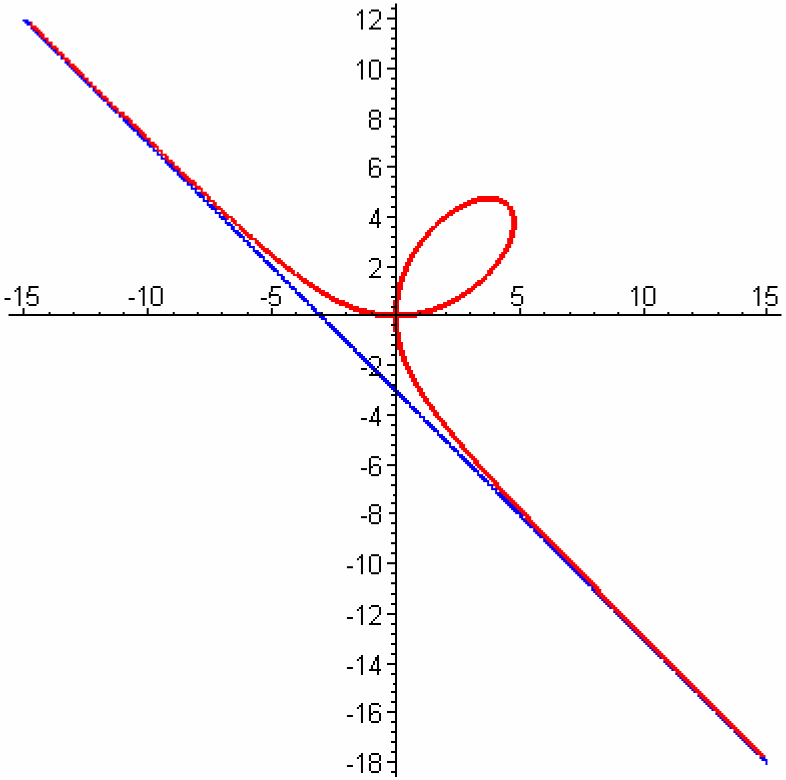
$$x^3 + y^3 - 3axy = 0$$

$$\rho = \frac{3a\sin\varphi\cos\varphi}{\sin^3\varphi + \cos^3\varphi}, \quad \varphi \in \left[0, \pi/2\right] \cup \left[3/4\,\pi, \pi\right] \cup \left[3/2\,\pi, 7/4\,\pi\right]$$

$$x = \frac{3at}{1+t^3}$$

$$y = \frac{3at^2}{1+t^3}$$

$$t \in R \setminus \{-1\}$$



# HELIX

Helix is the curve cutting the generators of a right circular cylinder under a constant angle $\beta$

$$x = a\cos t, \quad y = a\sin t, \quad z = at\cot\beta, \quad t \in [0, 4\pi]$$

$$\boxed{a = 2, \quad \beta = 78^o}$$