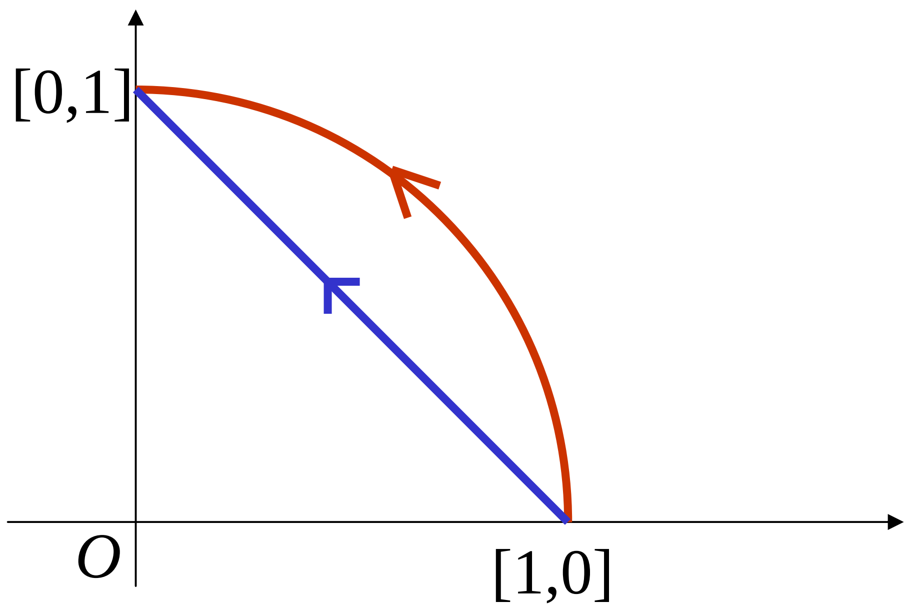
EXAMPLE

Calculate $\int_C \left(9x^2 - 2y + 3\right) dx + \left(-2x - 4\right) dy$

Where C is

1. The part of he circle $x^2 + y^2 = 1$that lies in the first quadrant oriented anti-clockwise

2. The line segment between the points [1,0] and [0,1]



1. $x^2+y^2=1$

$$\int_C \left(9x^2-2y+3\right)dx+\left(-2x-4\right)dy=\int_0^{\pi/2} -9\cos^2 t\sin t+2\sin^2 t-3\sin t-2\cos^2 t-4\cos t\,dt=$$

$$=\int_0^{\pi/2} -9\cos^2 t\sin t-2\cos 2t-3\sin t-4\cos t\;dt=-9\int_0^1 t^2\;dt-2\int_0^{\pi/2}\cos 2t\;dt-3-4=$$

$$=-9\left[\frac{t^3}{3}\right]_0^1-7=-10$$

2. $y=1-x$

$$-\int_C \left(9x^2-2y+3\right)dx+\left(-2x-4\right)dy=\int_0^1 \left(9x^2-2(1-x)+3\right)dx+\left(-2x-4\right)(-dx)=$$

$$=-\int_0^1 9x^2+4x+5\;dx=-\left[3x^3+2x^2+5x\right]_0^1=-10$$

The identical results of the above calculations are no coincidence.

The reason is that the vector components

$$(f_1(x,y),\ f_2(x,y)) = (9x^2 - 2y + 3, -2x - 4)$$

are the partial derivatives by x and y respectively of the function

$$F\,(x,y) = 3x^3 - 2xy + 3x - 4y.$$

Let $\boldsymbol{f}(x,y) = (f_1(x,y),\ f_2(x,y))$ be a vector field with $f_1(x,y),\ f_2(x,y)$ continuous in a planar area A containing a regular curve C given by the parametric equations

$$x = \varphi(t),\ y = \psi(t),\ t \in [a,b]$$

oriented in correspondence with these parametric equations. Let A denote the point $[\varphi(a),\psi(a)]$ and B the point $[\varphi(b),\psi(b)]$.

A function $F(x,y)$ in A such that

- $F(x,y)$ has continuous first partial derivatives

- $$f_1(x,y) = \frac{\partial}{\partial x}F(x,y), \quad f_2(x,y) = \frac{\partial}{\partial y}F(x,y)$$

is called the <u>***potential***</u> of the vector field $\boldsymbol{f}(x,y)$

Let the potential F (x,y) of $\vec{f}(x,y)$ exist and calculate the line integral of $\vec{f}(x,y)$ along C:

$$\int_C f_1(x,y)\,dx + f_2(x,y)\,dy = \int_a^b f_1(\varphi(t),\psi(t))\varphi'(t) + f_2(\varphi(t),\psi(t))\psi'(t) =$$

$$= \int_a^b \left(\frac{d}{dt}F(\varphi(t),\psi(t))\right)dt = [F(\varphi(t),\psi(t))]_a^b =$$

$$= F(\varphi(b),\psi(b)) - F(\varphi(a),\psi(a)) = F(B) - F(A).$$

This means that the result only depends on the value of $F(x,y)$ at A and B and is independent of the choice of curve C.

Clearly, a similar result can be obtained for 3D curves: If *F* (x,y,z) is a potential of $\vec{f}(x, y, z)$ so that

$$f_1(x, y, z) = \frac{\partial}{\partial x} F(x, y, z)$$

$$f_2(x, y, z) = \frac{\partial}{\partial y} F(x, y, z)$$

$$f_3(x, y, z) = \frac{\partial}{\partial z} F(x, y, z)$$

then

$$\boxed{\int_C f_1(x, y, z)dx + f_2(x, y, z)dy + f_3(x, y, z)dz = F(B) - F(B)}$$

for every curve *C* going from *A* to *B*.

Let $F(x,y,z)$ and $G(x,y,z)$ be two potentials of $\boldsymbol{f}(x,y,z)$.

Then

$$f_1(x,y,z) = \frac{\partial}{\partial x}F(x,y,z) \qquad f_1(x,y,z) = \frac{\partial}{\partial x}G(x,y,z)$$

$$f_2(x,y,z) = \frac{\partial}{\partial y}F(x,y,z) \qquad f_2(x,y,z) = \frac{\partial}{\partial y}G(x,y,z)$$

$$f_3(x,y,z) = \frac{\partial}{\partial z}F(x,y,z) \qquad f_3(x,y,z) = \frac{\partial}{\partial z}G(x,y,z)$$

$$\frac{\partial}{\partial x}\left(F(x,y,z) - G(x,y,z)\right) = 0$$

$$\frac{\partial}{\partial y}\left(F(x,y,z) - G(x,y,z)\right) = 0$$

$$\frac{\partial}{\partial z}\left(F(x,y,z) - G(x,y,z)\right) = 0$$

$G(x,y,z)=F(x,y,z)$ + const

Given the functions $f_1(x,y)$ and $f_2(x,y)$, how can we know that a function $F\,(x,y)$ exists such that

$$f_1(x, y) = \frac{\partial}{\partial x} F(x, y), \quad f_2(x, y) = \frac{\partial}{\partial y} F(x, y)$$

We have $\frac{\partial}{\partial y} f_1(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$ and $\frac{\partial}{\partial x} f_2(x,y) = \frac{\partial^2}{\partial y \partial x} F(x,y)$

From what we know about $F(x,y)$, it follows that

$$\frac{\partial^2}{\partial x \partial y} F(x,y) = \frac{\partial^2}{\partial y \partial x} F(x,y)$$

Thus we can conclude that

$$\boxed{\frac{\partial}{\partial y} f_1(x,y) = \frac{\partial}{\partial x} f_2(x,y)}$$

is a condition <u>necessary</u> for the existence of the function $F(x,y)$.

It can be proved that this condition is also <u>sufficient.</u>

In the event of a 3D curve and a function $F(x,y,z)$, if

$$f_1 = \frac{\partial}{\partial x} F \qquad f_2 = \frac{\partial}{\partial y} F \qquad f_3 = \frac{\partial}{\partial z} F$$

then clearly

$$\frac{\partial f_1}{\partial y} = \frac{\partial^2 F}{\partial x \partial y} \qquad \frac{\partial f_2}{\partial x} = \frac{\partial^2 F}{\partial y \partial x} \qquad \frac{\partial f_3}{\partial x} = \frac{\partial^2 F}{\partial z \partial x}$$

$$\frac{\partial f_1}{\partial z} = \frac{\partial^2 F}{\partial x \partial z} \qquad \frac{\partial f_2}{\partial z} = \frac{\partial^2 F}{\partial y \partial z} \qquad \frac{\partial f_3}{\partial y} = \frac{\partial^2 F}{\partial z \partial y}$$

This means that

$$\boxed{\frac{\partial f_3}{\partial y}=\frac{\partial f_2}{\partial z} \qquad \frac{\partial f_1}{\partial z}=\frac{\partial f_3}{\partial x} \qquad \frac{\partial f_2}{\partial x}=\frac{\partial f_1}{\partial y}} \qquad \textbf{(1)}$$

or

$$\boxed{\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}=0 \qquad \frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}=0 \qquad \frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}=0} \qquad \textbf{(2)}$$

are conditions necessary for the existence of the function $F(x,y)$.

Again, it can be proved that they are also sufficient.

Conditions (2) can also be expressed using the following formal calculation.

Suppose that, next to the vector

$$\vec{f}(x,y,z) = f_1(x,y,z)\vec{i} + f_2(x,y,z)\vec{j} + f_3(x,y,z)\vec{k}$$

we also have a formal vector

$$\boxed{\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}}$$

This vector is also called the ***nabla operator***.

Let us calculate the vector product

$$\vec{f}(x,y,z) \times \nabla$$

$$\vec{f}(x,y,z)\times\nabla = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} =$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\vec{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\vec{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\vec{k}$$

Thus the conditions (2) can be written as

$$\vec{f}(x,y,z)\times\nabla = \vec{0}$$

where $\vec{0}$ is the zero vector.

The vector field $\vec{f}(x, y, z) \times \nabla = \vec{0}$ is called the ***rotation or curl of the vector field*** $\vec{f}(x, y, z)$

If the rotation of a vector field is the zero vector in an area *A*, the vector field is said to be ***irrotational*** or conservative in *A*.

The line integral of a vector field is independent of the curve along which the endpoint is reached from the starting point exactly if the vector field is irrotational.

To find a potential to functions $f_1(x,y,z)$, $f_2(x,y,z)$, $f_3(x,y,z)$, first we must verify that

$$\vec{f}(x,y,z)\times\nabla=\vec{0} \qquad \text{or} \qquad \frac{\partial f_3}{\partial y}=\frac{\partial f_2}{\partial z} \quad \frac{\partial f_1}{\partial z}=\frac{\partial f_3}{\partial x} \quad \frac{\partial f_2}{\partial x}=\frac{\partial f_1}{\partial y}$$

and then perform the following calculations:

$$F(x,y,z)=\int f_1(x,y,z)dx+G(y,z)$$

$$\frac{\partial}{\partial y}G(y,z)=\frac{\partial}{\partial y}F(x,y,z)-\int\left\{\frac{\partial}{\partial y}f_1(x,y,z)\right\}dx=F_2(y,z)$$

$$G(y,z)=\int F_2(y,z)\,dy+H(z)$$

$$\frac{d}{dz}H(z)=\frac{\partial}{\partial z}G(y,z)-\int\left\{\frac{\partial}{\partial z}F_2(y,z)\right\}dy=F_3(z)$$

$$H(z)=\int F_3(z)\,dz+\text{const}$$

$$\boxed{F(x,y,z)=\int f_1(x,y,z)\,dx+\int F_2(y,z)\,dy+\int F_3(z)\,dz+\text{const}}$$

Example

Find a potential to the vector field

$$\overrightarrow{f}(x,y,z)=\left(3x^2yz-2y^3\right)\vec{i}+\left(x^3z-6xy^2+2yz^2\right)\vec{j}+\left(x^3y+2y^2z\right)\vec{k}$$

Solution

We have

$$\frac{\partial}{\partial z}\left(x^3z-6xy^2+2yz^2\right)=x^3+4yz=\frac{\partial}{\partial y}\left(x^3y+2y^2z\right)$$

$$\frac{\partial}{\partial x}\left(x^3y+2y^2z\right)=3x^2y=\frac{\partial}{\partial z}\left(3x^2yz-2y^3\right)$$

$$\frac{\partial}{\partial y}\left(3x^2yz-2y^3\right)=3x^2z-6y^2=\frac{\partial}{\partial x}\left(x^3z-6xy^2+2yz^2\right)$$

$$F(x,y,z)=\int 3x^2yz-2y^3\,dx = x^3yz-2xy^3+G(y,z)$$

$$\frac{\partial}{\partial y}G(y,z)=x^3z-6xy^2+2yz^2-\left(x^3z-6xy^2\right)=2yz^2$$

$$G(y,z)=y^2z^2+H(z)$$

$$F(x,y,z)=x^3yz-2xy^3+y^2z^2+H(z)$$

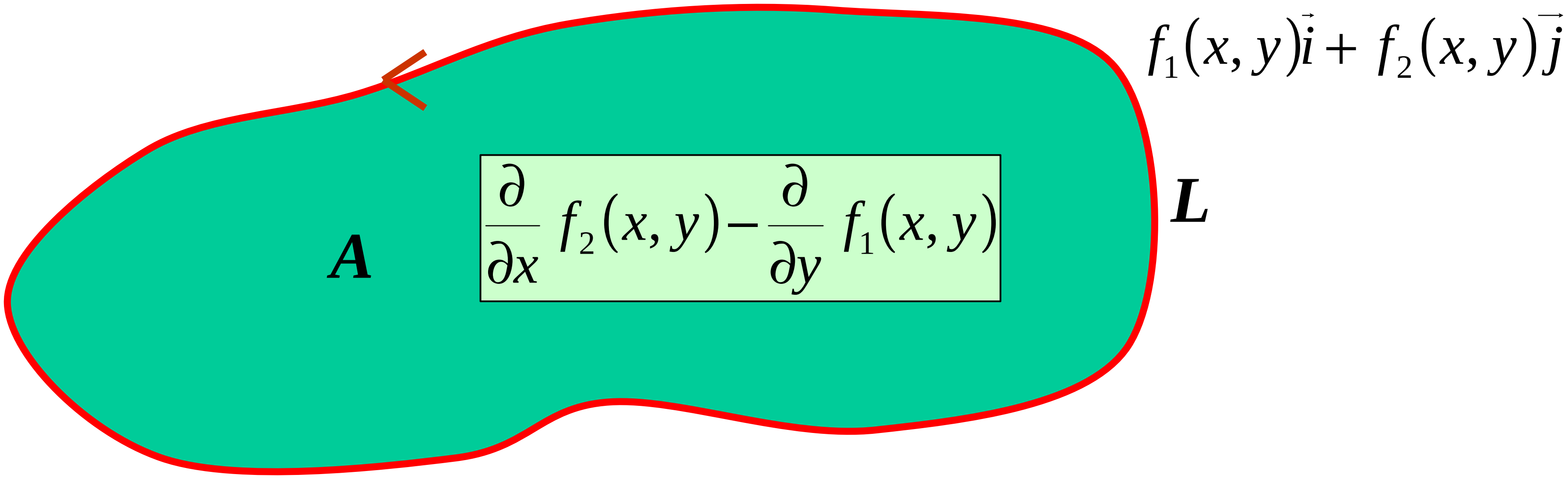
$$\frac{d}{dz}H(z)=\left(x^3y+2y^2z\right)-\left(x^3y+2y^2z\right)=0$$

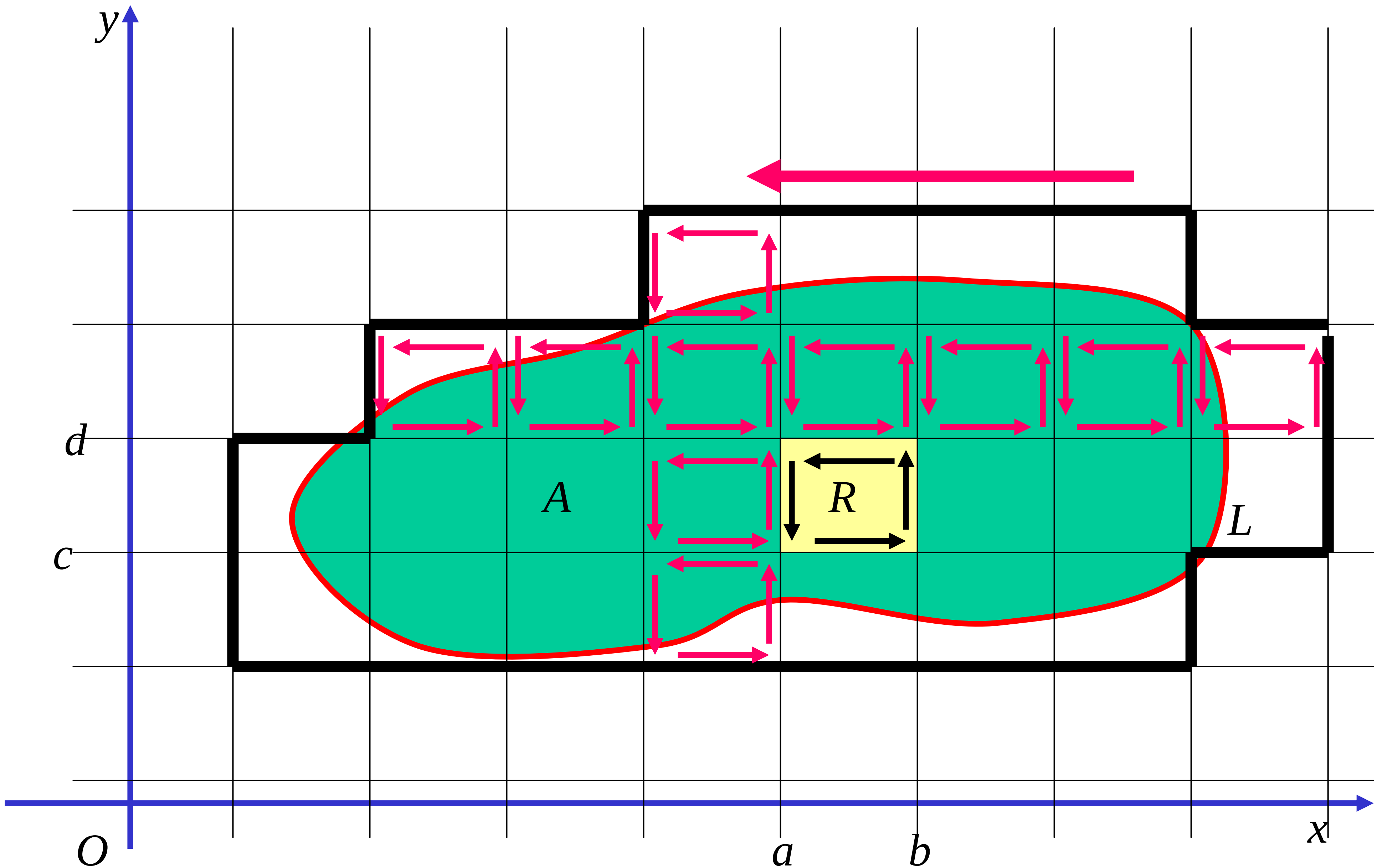
$$\boxed{F(x,y,z)=x^3yz-2xy^3+y^2z^2+\text{const}}$$

GREEN'S THEOREM

Let $f(x,y) = f_1(x,y)\,\boldsymbol{i} + f_2(x,y)\,\boldsymbol{j}$ be a vector field in a planar area M with continuous first order partial derivatives and let L be a closed regular curve in M oriented anticlockwise. Let A denote the planar area bounded by L. Then

$$\boxed{\oint_L f_1(x,y)dx + f_2(x,y)dy = \iint_A \left(\frac{\partial}{\partial x} f_2(x,y) - \frac{\partial}{\partial y} f_1(x,y)\right) dxdy}$$



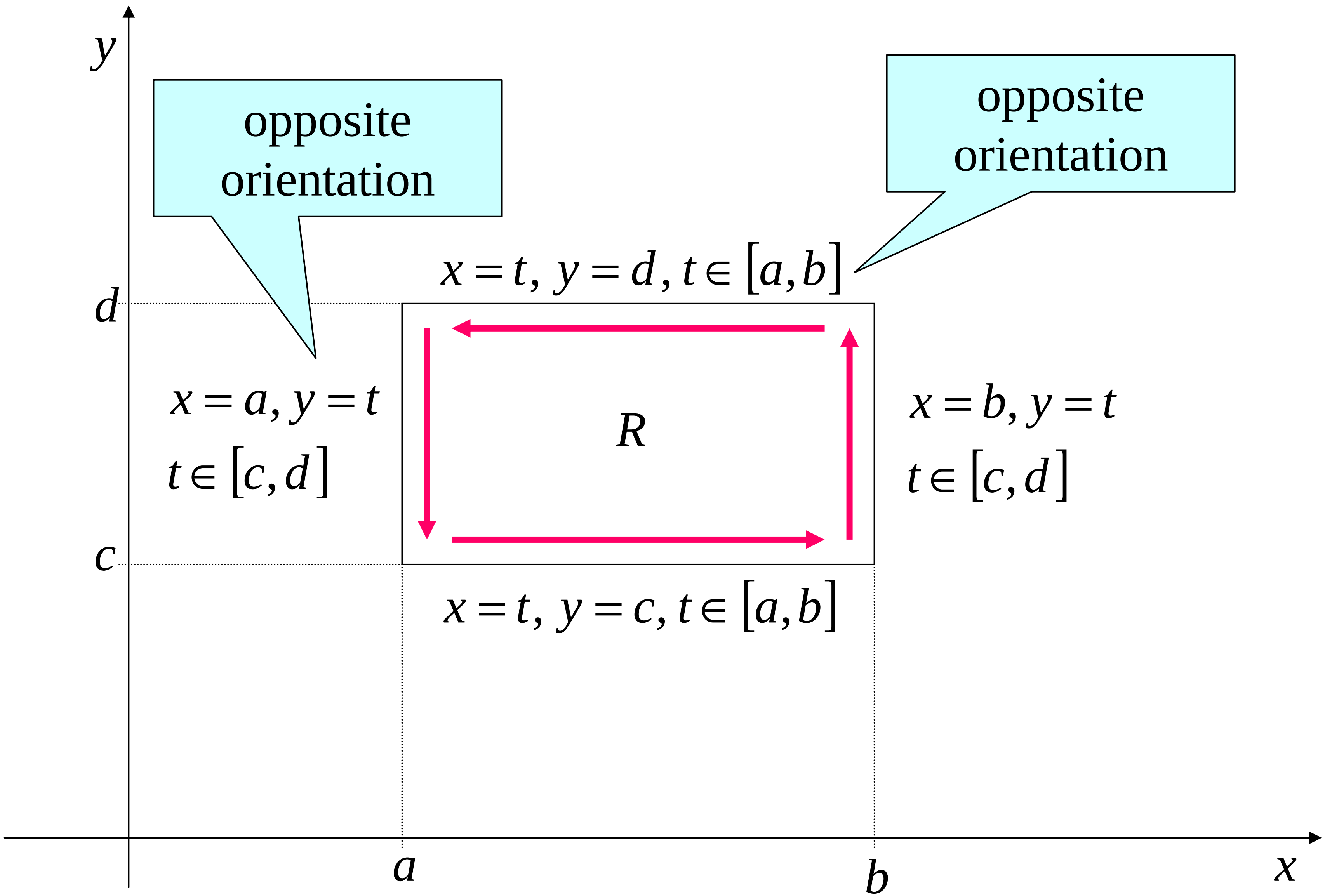


y
x
O
a
b
c
d
A
R
L

NOTE

From the picture it is clear that the line integrals round all the rectangles of the grid covering the area A add up to the line integral round the thick contour covering A since the line integrals round the inner sides of the rectangle grid are always calculated twice – each time with a different sign.

We will now calculate the line integral round the yellow rectangle R in the picture. The line integrals round other rectangles of the grid covering A would be calculated in the same way.



opposite orientation
opposite orientation
$x = t,\ y = d,\ t \in [a,b]$
$x = a, y = t$
$t \in [c,d]$
R
$x = b, y = t$
$t \in [c,d]$
$x = t,\ y = c,\ t \in [a,b]$
y
d
c
a
b
x

$$I_R = \oint_R f_1(x,y)\,dx + f_2(x,y)\,dy = I_1 + I_2 + I_3 + I_4 \quad \text{where}$$

$$I_1 = \int_a^b f_1(x,c)\,dx \qquad\qquad I_2 = \int_c^d f_2(b,y)\,dy$$

$$I_3 = -\int_a^b f_1(x,d)\,dx \qquad\qquad I_4 = -\int_c^d f_2(a,y)\,dy$$

$$I_R = \int_a^b \left(f_1(x,c) - f_1(x,d) \right) dx + \int_c^d \left(f_2(b,y) - f_2(a,y) \right) dy$$

$$I_R = \int_c^d \left(f_2(b, y) - f_2(a, y) \right) dy - \int_a^b \left(f_1(x, d) - f_1(x, c) \right) dx$$

Using the Lagrange theorem for both integrals, we can write

$$I_R = (b-a) \int_c^d \left(\frac{\partial}{\partial x} f_2(\beta_x, y) \right) dy - (d-c) \int_a^b \left(\frac{\partial}{\partial y} f_1(x, \alpha_y) \right) dx$$

where $\mathrm{c} < \beta_x < d$ and $a < \alpha_y < b$

Finally, using the mean value theorem for integrals, we get

$$I_R = (b-a)(d-c) \frac{\partial}{\partial x} f_2(\beta_x, \beta_y) - (d-c)(b-a) \frac{\partial}{\partial y} f_1(\alpha_x, \alpha_y)$$

where $\mathrm{c} < \beta_y < d$ and $a < \alpha_x < b$

If we calculate I_R for every rectangle of the grid covering A and add them up, we get, on the left-hand side of the equation, the line integral of the vector field $f_1(x,y)\,\boldsymbol{i} + f_2(x,y)\,\boldsymbol{j}$ round the thick contour in the picture (see the note) and, on the right-hand side, an integral sum lying between the lower and upper integral sums of the double integral

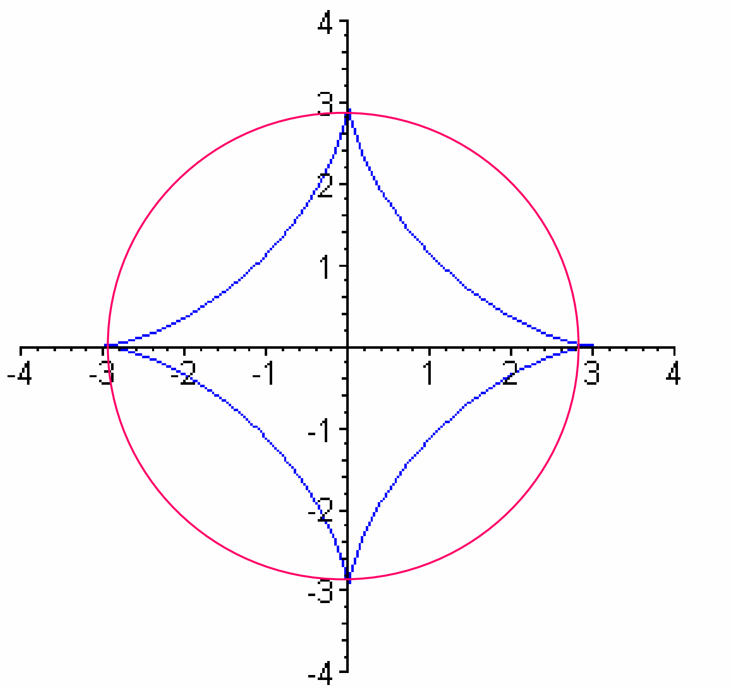
$$\iint_A \left(\frac{\partial}{\partial x} f_2(x,y) - \frac{\partial}{\partial y} f_1(x,y) \right) dxdy$$

Thus, when the norm of the grid tends to zero, this equation tends to what Green's theorem says.

EXAMPLE

Calculate the area of the astroid given by the equations

$$x = a\cos^3 t,\ y = a\sin^3 t,\ t \in [0, 2\pi]$$



For comparison, the circumscribed red circle has a radius of *a*

By Green's theorem, we have

$$\iint_A 1\,dxdy = \frac{1}{2}\oint_L -y\,dx + x\,dy$$

$$\oint_L -y\,dx + x\,dy = \int_0^{2\pi} -a\sin^3 t a 3\cos^2 t(-\sin t) + a\cos^3 t a 3\sin^2 t\cos t\,dt =$$

$$= 3a^2\int_0^{2\pi}\sin^4 t\cos^2 t + \cos^4 t\sin^2 t\,dt = 3a^2\int_0^{2\pi}\sin^2 t\cos^2 t\,dt =$$

$$= \frac{3a^2}{4}\int_0^{2\pi}(\sin 2t)^2\,dt = \frac{3a^2}{4}\int_0^{2\pi}\frac{1-\cos 4t}{2} = \frac{3\pi a^2}{4}$$

Thus the area of the astroid is $\frac{3\pi a^2}{8}$