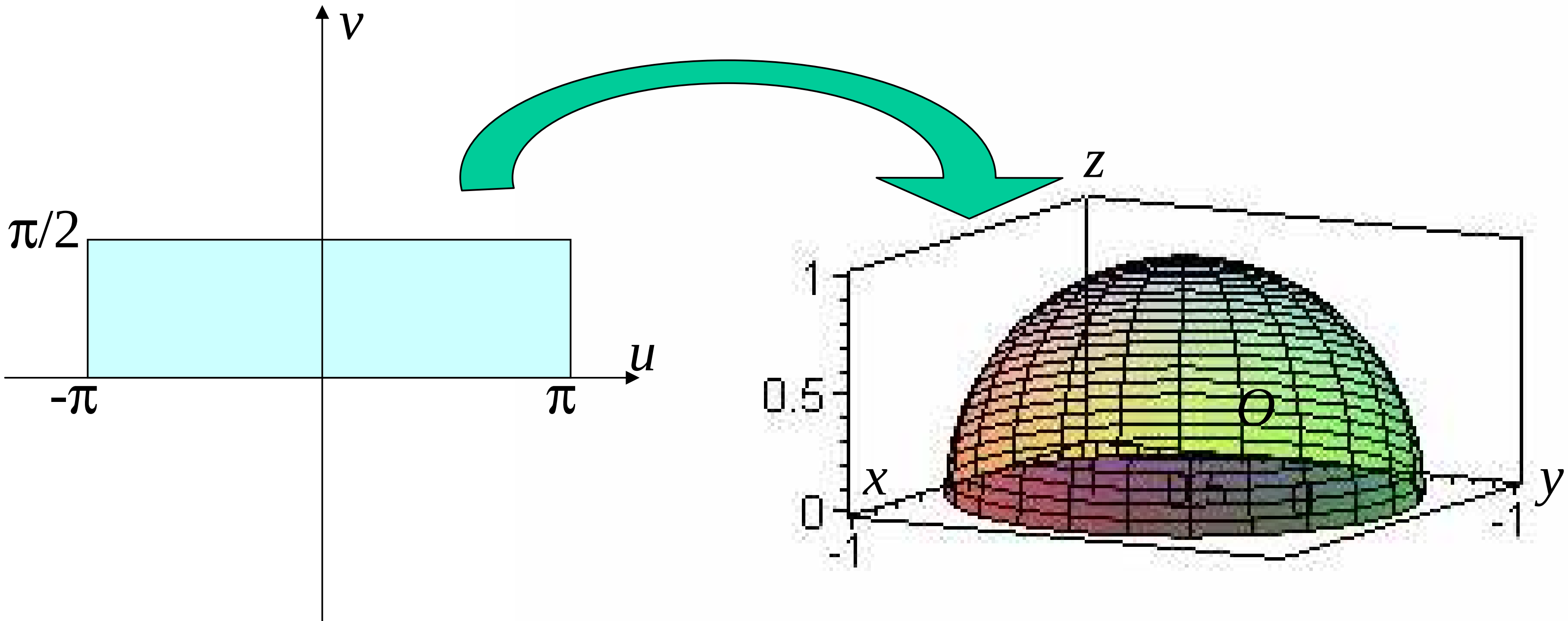
PARAMETRIC EQUATIONS OF A 3D SURFACE



$$x = \cos u \sin v$$

$$y = \sin u \sin v \quad [u, v] \in [-\pi, \pi] \times \left[0, \frac{\pi}{2}\right]$$

$$z = \cos v$$

SIMPLE 3D-SURFACES

- M is a planar area bounded by a closed regular curve ∂M

- $\varphi(u,v)$, $\psi(u,v)$, $\chi(u,v)$ are one-to-one mappings of M into R^3

- $\varphi(u,v)$, $\psi(u,v)$, $\chi(u,v)$ have continuous first partial derivatives on M

We will call the set

$$S=\{\ [x,y,z] \mid x = \varphi(u,v),\ \ y = \psi(u,v),\ \ z = \chi(u,v)\ \}$$

a ***<u>simple 3D-surface</u>***

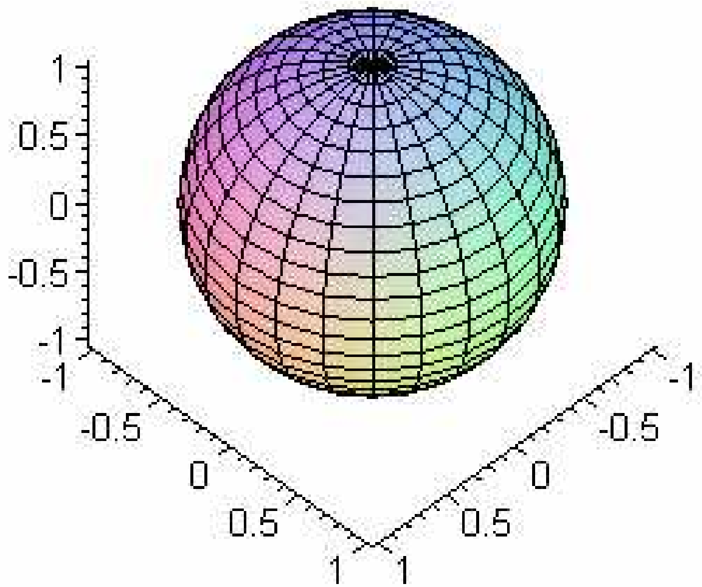
A 3D-surface S is closed if it divides R^3 into at least two contiguous parts that cannot be connected by a continuous line without crossing S.

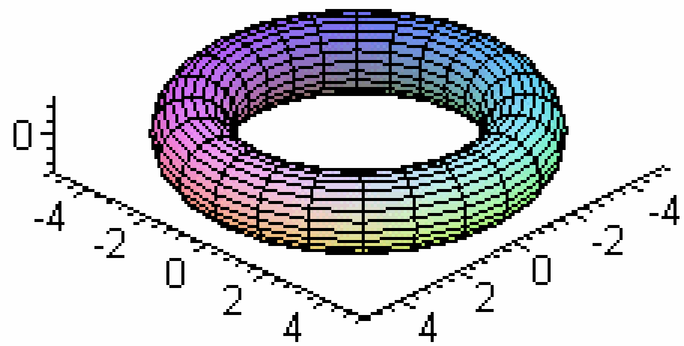
SPHERE

$$x = \cos u \sin v$$
$$y = \sin u \sin v \quad [u,v] \in [0,2\pi]\times[0,\pi]$$
$$z = \cos v$$

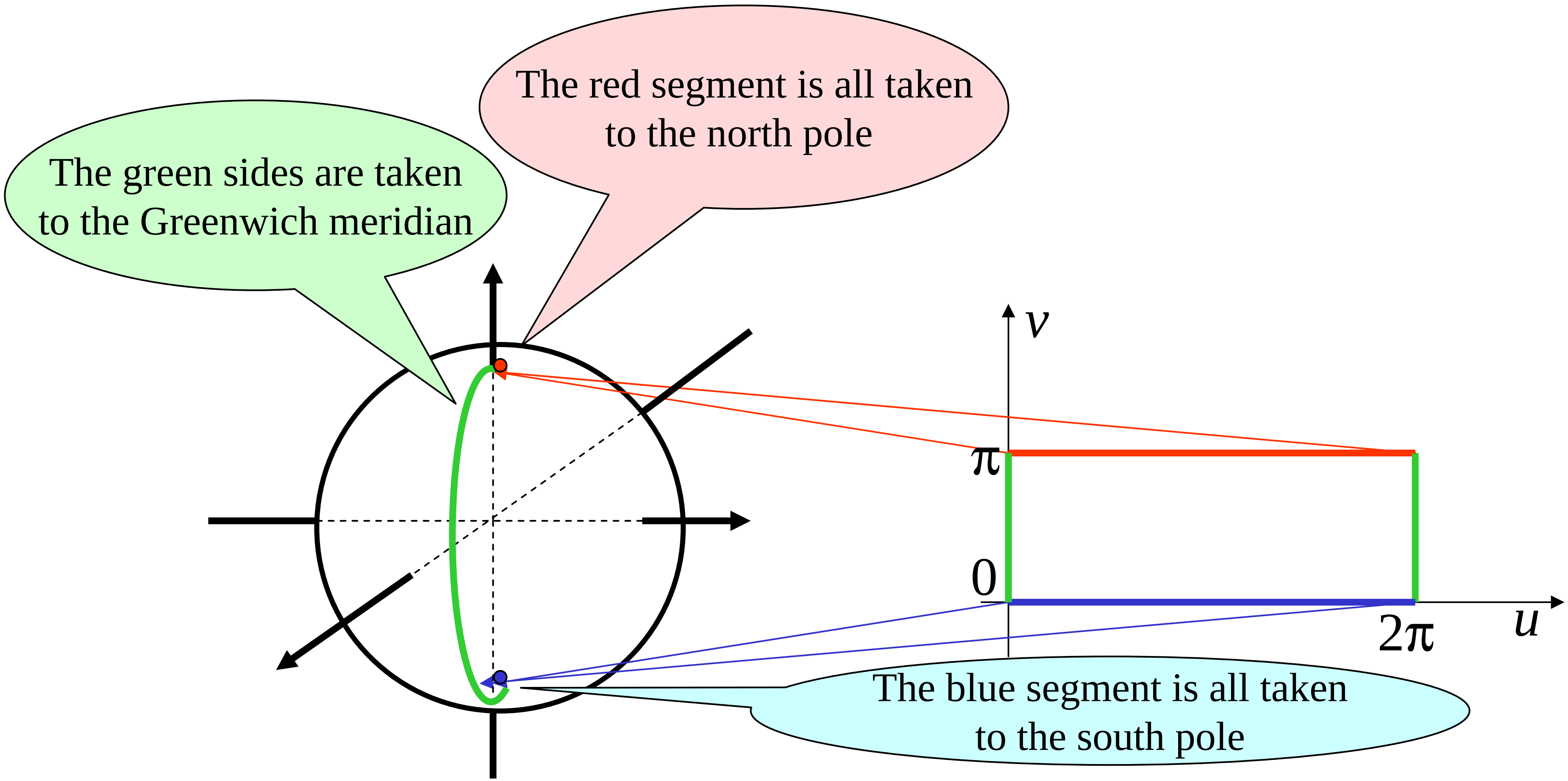
TORUS

$$x = \cos v(4+\cos u)$$
$$y = \sin v(4+\cos u) \quad [u,v] \in [0,2\pi]\times[0,2\pi]$$
$$z = \sin u$$



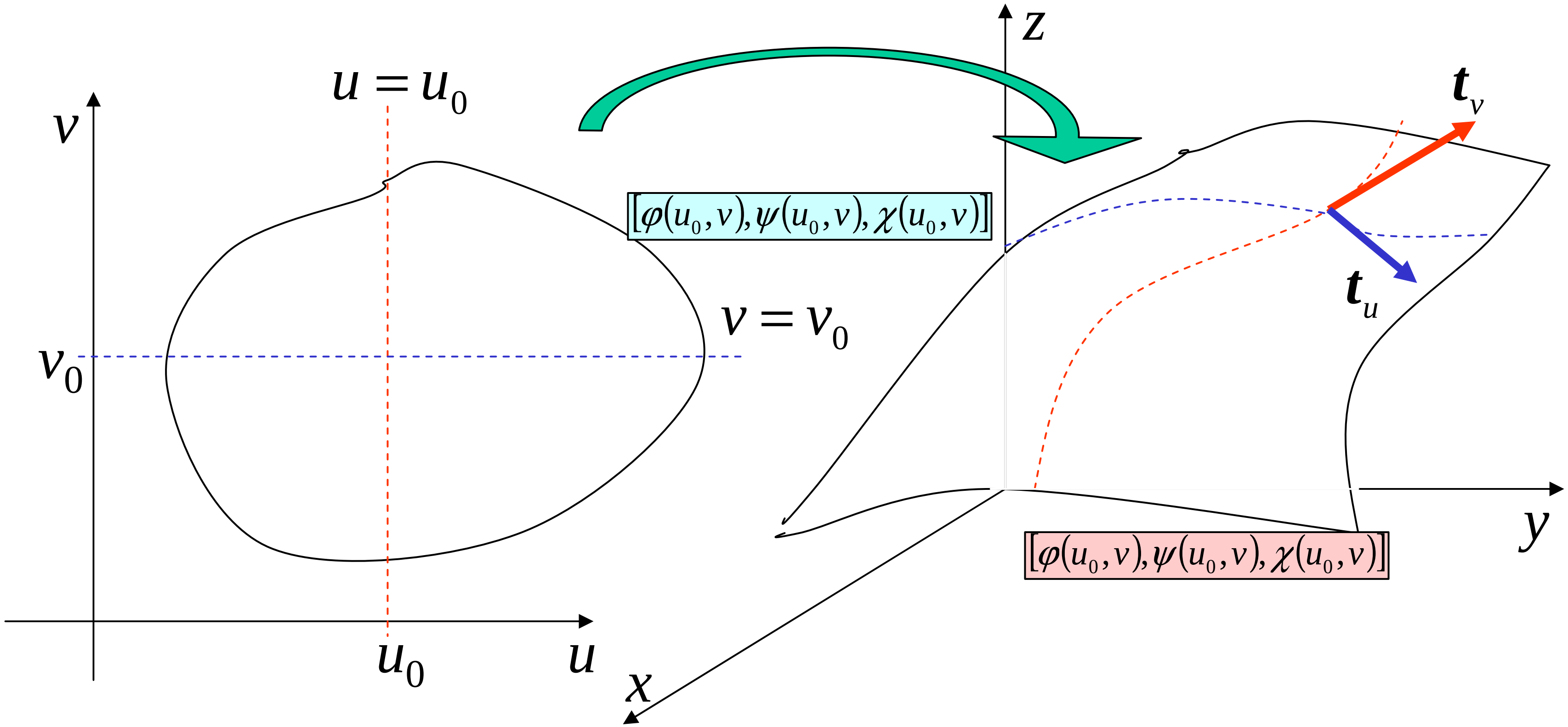


If we require the mappings $\varphi(u,v)$, $\psi(u,v)$, $\chi(u,v)$ to be one-to-one only on $M - \partial M$, that is, not including the boundary, the surfaces may also be closed. For example, with the sphere in the previous picture,



Tangent vectors

$$x=\varphi(u,v),\ y=\psi(u,v),\ z=\chi(u,v)$$

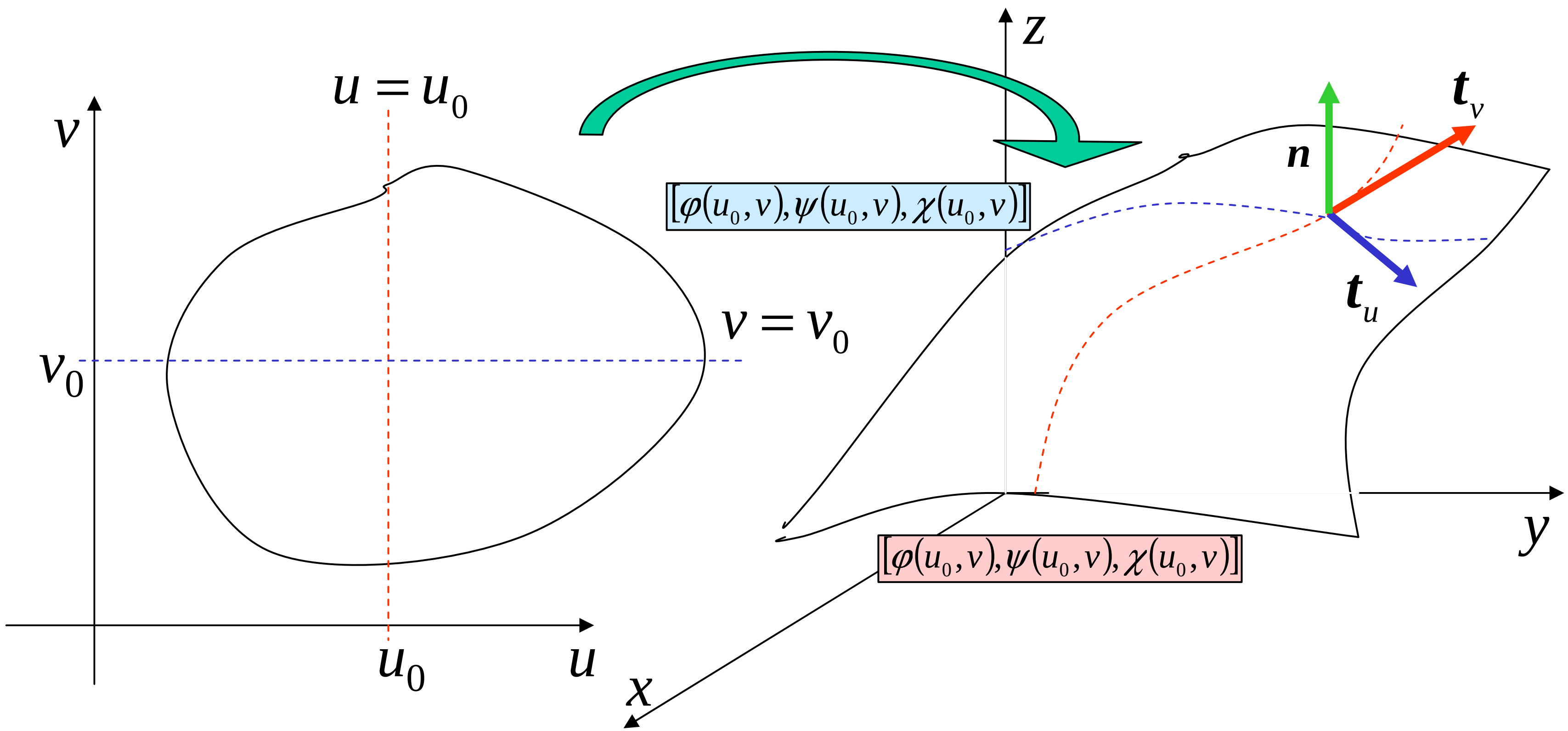


$$\boxed{\boldsymbol{t}_u=(\varphi'_u(u_0,v_0),\psi'_u(u_0,v_0),\chi'_u(u_0,v_0))}$$

$$\boxed{\boldsymbol{t}_v=(\varphi'_v(u_0,v_0),\psi'_v(u_0,v_0),\chi'_v(u_0,v_0))}$$

Normal

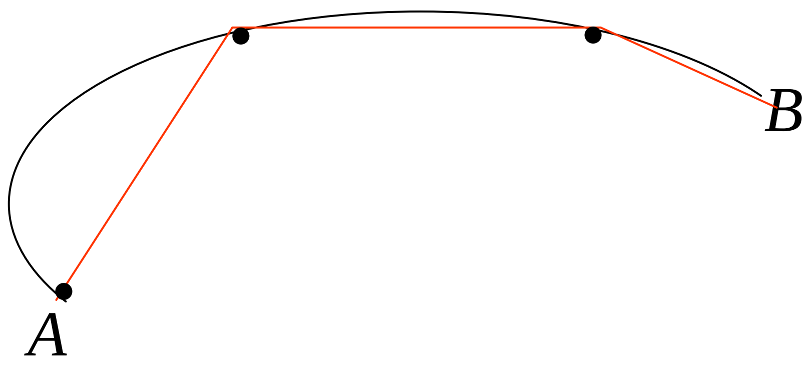
$$x=\varphi(u,v),\ y=\psi(u,v),\ z=\chi(u,v)$$

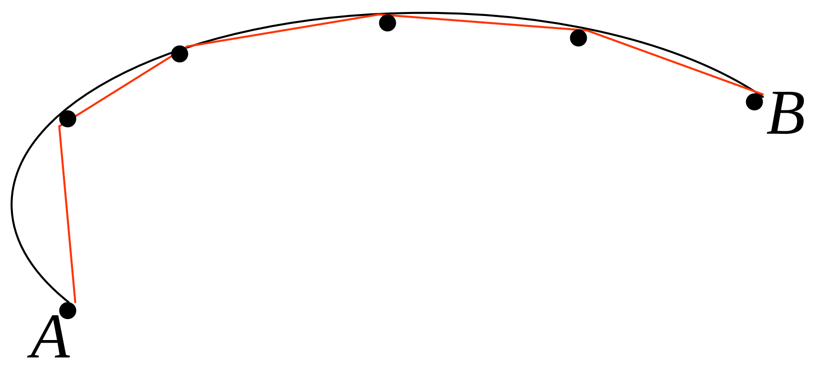


$$\boxed{\boldsymbol{n}=\boldsymbol{t}_u\times\boldsymbol{t}_v=\left(\psi'_u\chi'_v-\psi'_v\chi'_u,\chi'_u\varphi'_v-\chi'_v\varphi'_u,\varphi'_u\psi'_v-\varphi'_v\psi'_u\right)}$$

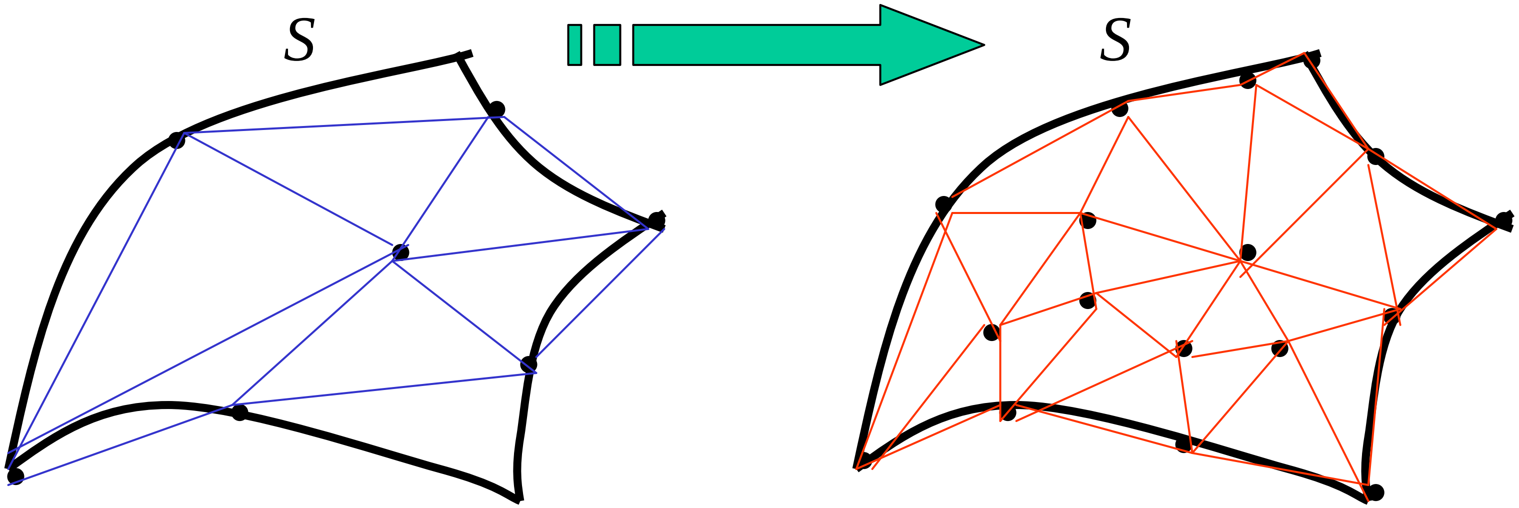
AREA OF 3D-SURFACES

To define the area of a 3D surface we could try to proceed in a way similar to defining the length of a curve. For a curve, we took all polygons inscribed into it and defined the length as the lub of the lengths of all such polygons. This could be achieved by letting the norm of the polygons tend to zero.

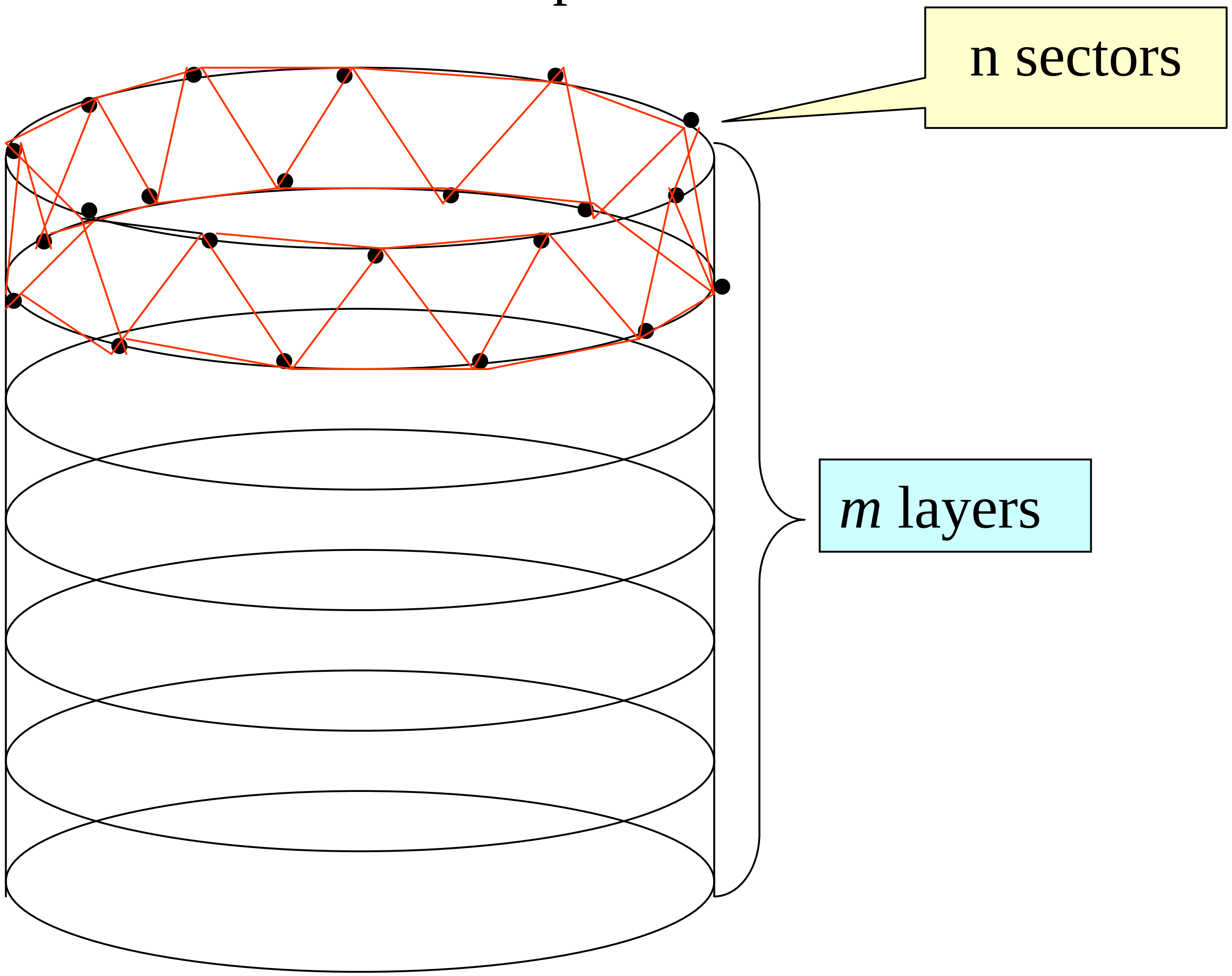


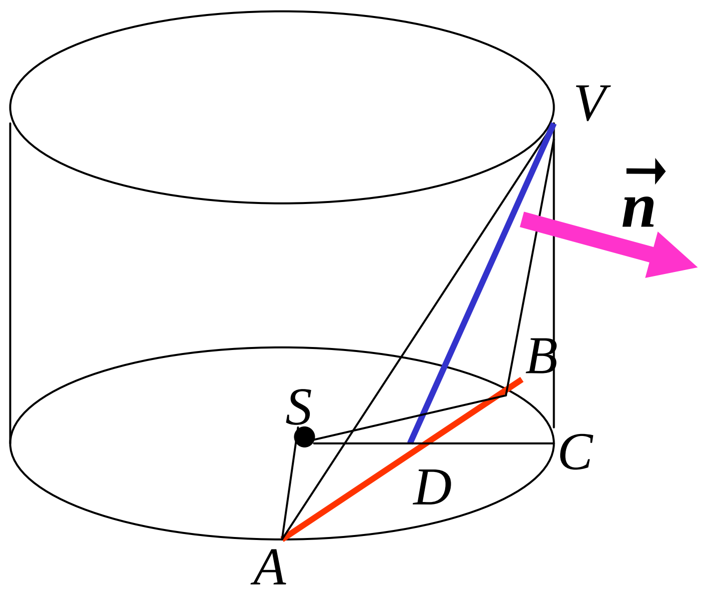


For a 3D-surface *S*, we could similarly take a triangulation consisting of triangles inscribed in *S* and let its norm, that is, the area of the largest triangle, tend to zero. The limit of the sum of the inscribed triangles would then be taken for the area of S.



Surprisingly, this is not possible even for such relatively simple surfaces as a cylindrical surface. The following example due to Schwarz illustrates this point





$$VC = \frac{h}{m}$$

$$SA = SB = SC = r$$

$$\angle BSC = \angle CSA = \frac{\pi}{n}$$

$$|ABV| = \overline{AD}\,\overline{DV}$$

$$AD = SA \sin\angle CSA = r\sin\frac{\pi}{n}$$

$$\overline{DV} = \sqrt{\overline{DC}^2 + \overline{VC}^2}$$

$$\overline{DC} = \overline{SC} - \overline{SD} = r - r\cos\frac{\pi}{n}$$

$$\overline{DV} = \sqrt{r^2\left(1-\cos\frac{\pi}{n}\right)^2 + \frac{h^2}{m^2}}$$

$$|ABV| = r\sin\frac{\pi}{n}\sqrt{r^2\left(1-\cos\frac{\pi}{n}\right)^2 + \frac{h^2}{m^2}}$$

The area S of the cylinder surface:

$$S = \lim_{\substack{m\to\infty \\ n\to\infty}} 2mn\left|ABV\right| =$$

$$= \lim_{\substack{m\to\infty \\ n\to\infty}} 2mnr\sin\frac{\pi}{n}\sqrt{r^2\left(1-\cos\frac{\pi}{n}\right)^2+\frac{h^2}{m^2}} =$$

$$= \lim_{\substack{m\to\infty \\ n\to\infty}} 2nr\sin\frac{\pi}{n}\sqrt{r^2m^2\left(1-\cos\frac{\pi}{n}\right)^2+h^2} =$$

$$= \lim_{\substack{m\to\infty \\ n\to\infty}} 2\pi r\,\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\sqrt{r^2m^2\left(1-\cos\frac{\pi}{n}\right)^2+h^2} = \lim_{\substack{m\to\infty \\ n\to\infty}} 2\pi r\sqrt{r^2m^2\left(1-\cos\frac{\pi}{n}\right)^2+h^2}$$

However, the last limit does not exist.

Indeed, since *m* and *n* may tend to infinity in an arbitrary way, we can assume that $\lim\limits_{\substack{m\to\infty \\ n\to\infty}} \frac{m}{n^2} = q$

for some constant *q*. This means that, for large values of *n*, we can replace *m* by qn^2 in the limit:

$$\lim_{\substack{m\to\infty \\ n\to\infty}} 2\pi r \sqrt{r^2 m^2 \left(1-\cos\frac{\pi}{n}\right)^2 + h^2} = \lim_{n\to\infty} 2\pi r \sqrt{r^2 q^2 n^4 \left(1-\cos\frac{\pi}{n}\right)^2 + h^2} =$$

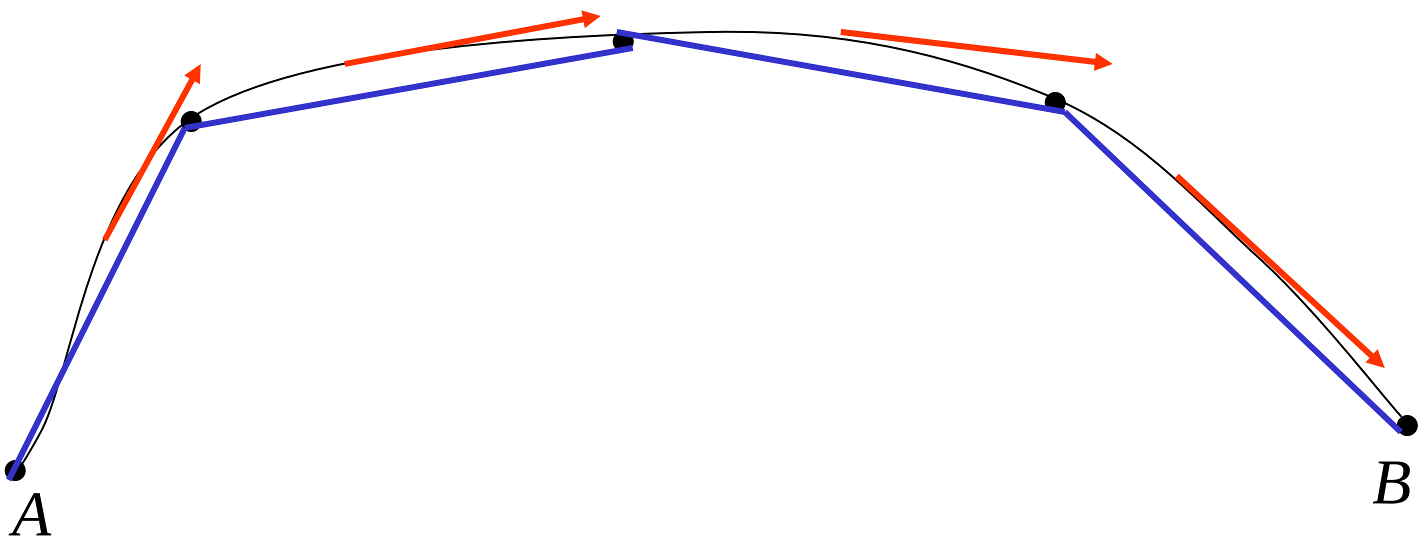
$$= \lim_{n\to\infty} 2\pi r \sqrt{r^2 q^2 n^4 \left(1-\cos\frac{\pi}{n}\right)^2 + h^2} = \lim_{n\to\infty} 2\pi r \sqrt{\frac{r^2 q^2 \pi^4 \sin^4 \frac{\pi}{n}}{\left(1+\cos\frac{\pi}{n}\right)^2 \frac{\pi^4}{n^4}} + h^2}$$

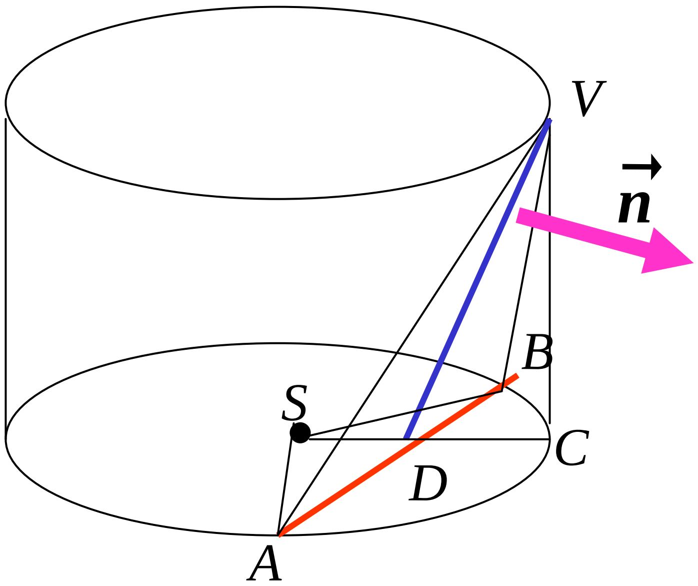
This means that

$$S = \lim_{n\to\infty} 2\pi r \sqrt{\frac{r^2 q^2 \pi^4 \sin^4 \frac{\pi}{n}}{\left(1+\cos\frac{\pi}{n}\right)^2 \frac{\pi^4}{n^4}} + h^2} = 2\pi r \sqrt{\frac{r^2 q^2 \pi^4}{4} + h^2}$$

and so the result depends on q. Thus the surface cannot be determined using this method.

The reason why this method fails when calculating surface areas whereas it is successful in calculating the length of a curve is the following. As can be seen in the picture below, with the length of the polygon segments becoming smaller, their direction tends to that of the appropriate tangent vectors:



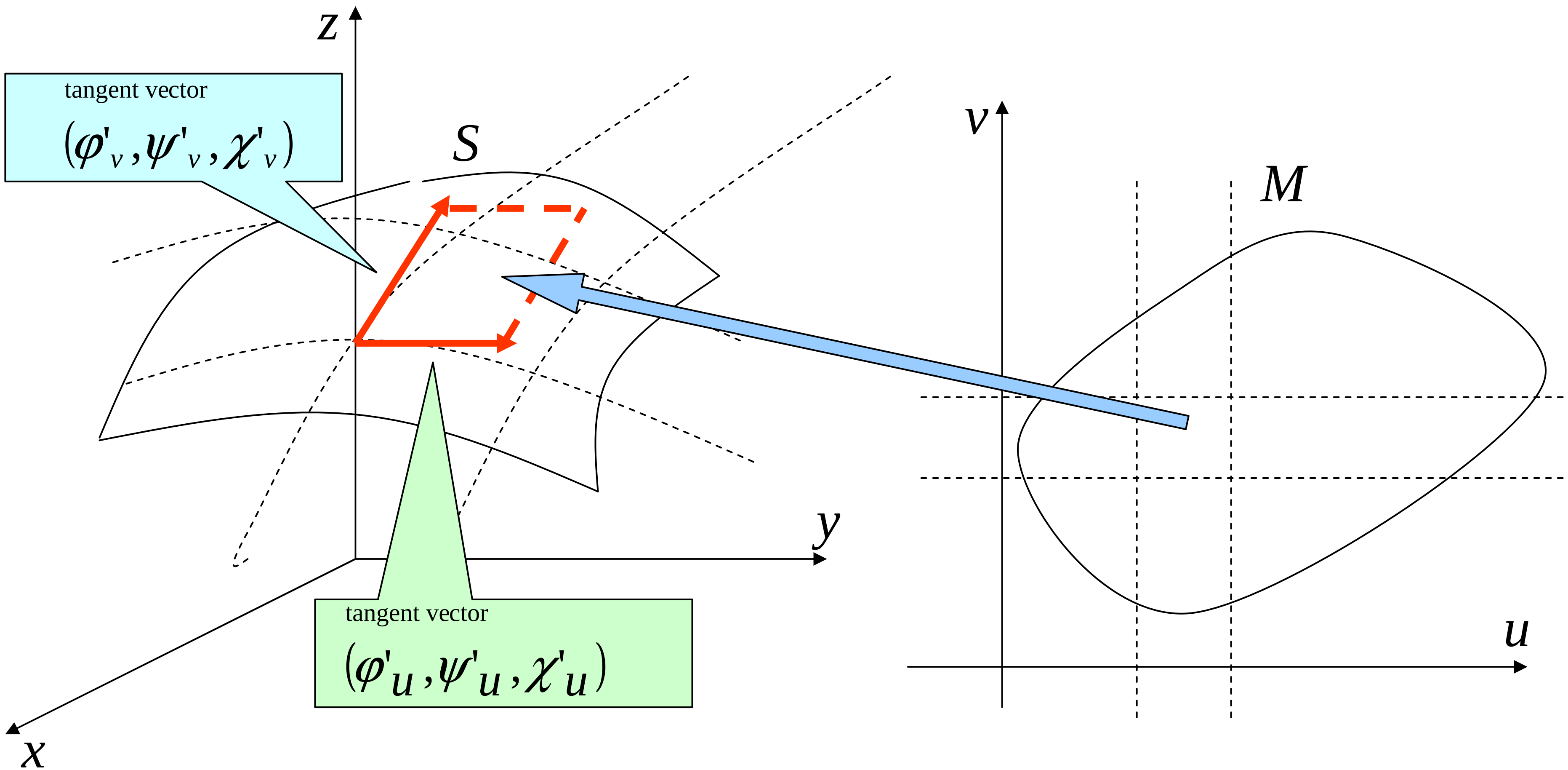


However, in Schwartz's example, the unit normals of the approximating triangles do not tend to those of the cylindrical surface. For example for the triangle on the left, the coordinates of the unit normal are:

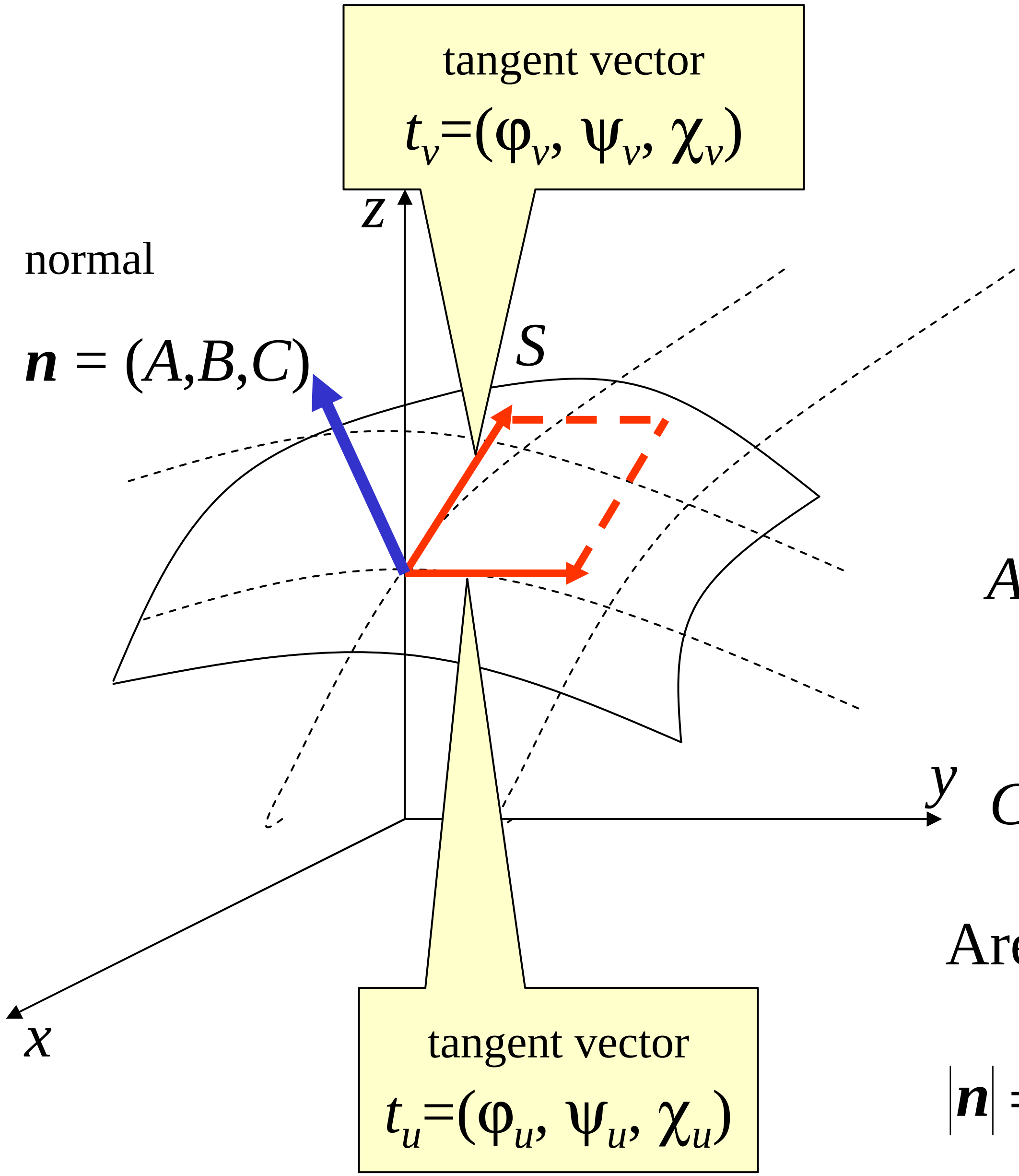
$$\vec{n} = \frac{2}{\sqrt{4+\pi^4 r^2 q^2}}\vec{i} + 0\vec{j} + \frac{-r\pi^2 q}{\sqrt{4+\pi^4 r^2 q^2}}\vec{k} \qquad \text{where} \qquad \lim_{\substack{m\to\infty \\ n\to\infty}} \frac{m}{n^2} = q.$$

Only for $q = 0$ do we get $\vec{n} = 1\vec{i} + 0\vec{j} + 0\vec{k}$, which is the unit normal of the cylindrical surface. However, for $q \to \infty$, we have $\vec{n} = 0\vec{i} + 0\vec{j} - 1\vec{k}$ which is perpendicular to it.

To calculate the area of a surface we will use the concept of a tangent plane.



the area of the piece will be approximated by that of the “peeling off” red tile made from the tangent plane



$$\boldsymbol{n} = (A, B, C) = t_u \times t_v$$

$$t_u \times t_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \varphi_u & \psi_u & \chi_u \\ \varphi_v & \psi_v & \chi_v \end{vmatrix}$$

$$A = \begin{vmatrix} \psi_u & \chi_u \\ \psi_v & \chi_v \end{vmatrix} \qquad B = \begin{vmatrix} \chi_u & \varphi_u \\ \chi_v & \varphi_v \end{vmatrix}$$

$$C = \begin{vmatrix} \varphi_u & \psi_u \\ \varphi_v & \psi_v \end{vmatrix}$$

Area of the tile = $|\boldsymbol{n}|$

$$|\boldsymbol{n}| = \sqrt{A^2 + B^2 + C^2}$$

The area |S| of a surface S defined by the parametric equations

$$x = \varphi(u,v),\ y = \psi(u,v),\ z = \chi(u,v), \quad [u,v] \in M$$

can be calculated using the following formula

$$|S| = \iint_M \sqrt{\begin{vmatrix} \psi'_u & \chi'_u \\ \psi'_v & \chi'_v \end{vmatrix}^2 + \begin{vmatrix} \chi'_u & \varphi'_u \\ \chi'_v & \varphi'_v \end{vmatrix}^2 + \begin{vmatrix} \varphi'_u & \psi'_u \\ \varphi'_v & \psi'_v \end{vmatrix}^2}\ du\, dv$$

For a 3D-surface *S* expressed by the explicit function

$$z = f(x, y), \quad [x, y] \in M$$

we get the following formula

$$|S| = \iint_M \sqrt{\begin{vmatrix} 0 & z'_u \\ 1 & z'_v \end{vmatrix}^2 + \begin{vmatrix} z'_u & 1 \\ z'_v & 0 \end{vmatrix}^2 + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^2} \, du \, dv$$

which yields

$$|S| = \iint_M \sqrt{z'^2_u + z'^2_v + 1} \, du \, dv$$

Example

Calculate the area of a sphere with a radius of *r*.

We will choose a sphere with the centre at [0,0,0],

which has the parametric equations

$$
\begin{aligned}
x &= r\cos u \sin v \\
y &= r\sin u \sin v \qquad [u,v] \in [0,2\pi]\times[0,\pi] \\
y &= r\cos v
\end{aligned}
$$

Clearly, for reasons of symmetry, we can consider an

M' : $[0,\pi/2]\times[0,\pi/2]$ multiplying the result by 8.

We have

$$\boldsymbol{t}_u = (-r\sin u \sin v, r\cos u \sin v, 0)$$

$$\boldsymbol{t}_v = (r\cos u \cos v, r\sin u \cos v, -r\sin v)$$

and so

$$A = -r^2 \cos u \sin^2 v,\ B = -r^2 \sin u \sin^2 v,\ C = -r^2 \sin^2 u \sin v \cos v$$

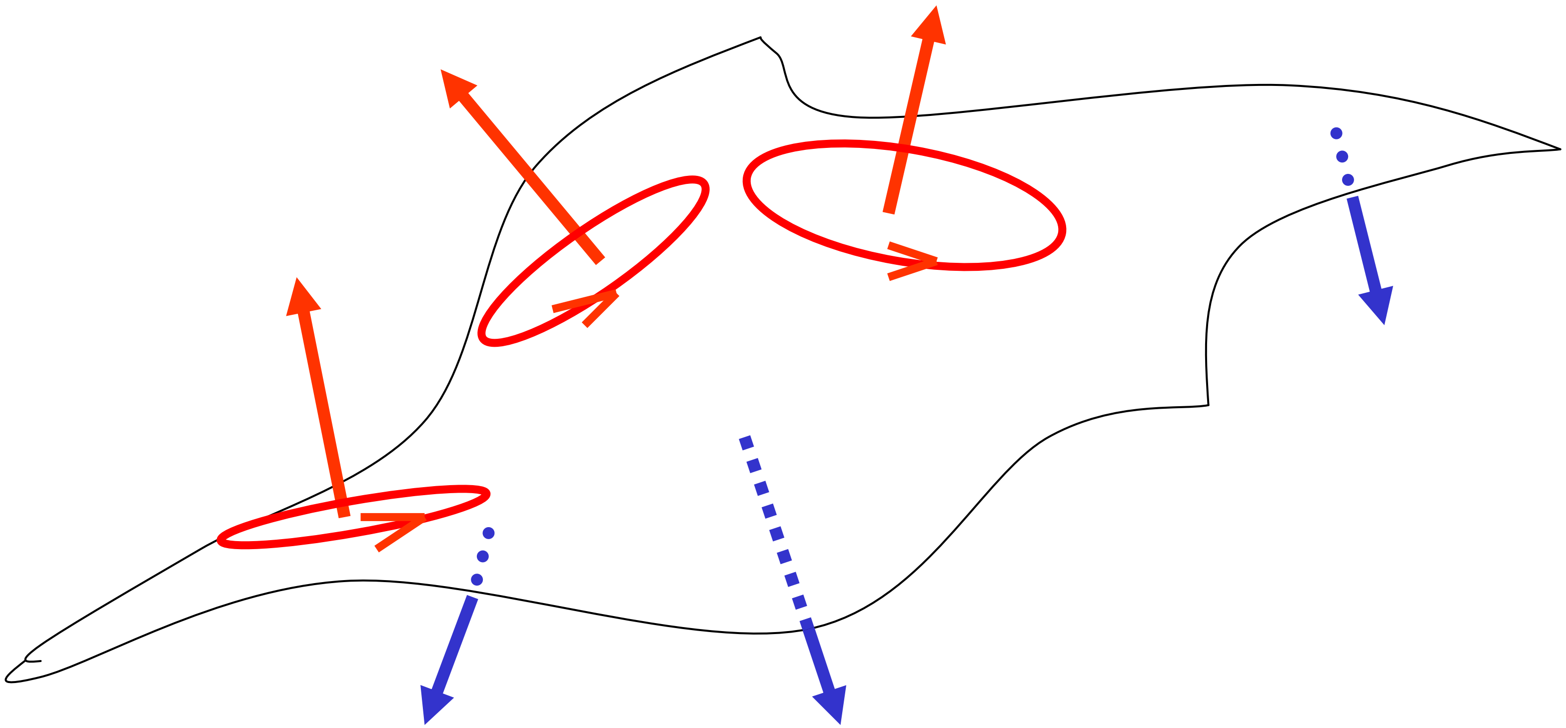
which yields

$$A^2 + B^2 + C^2 = r^4\left(\sin^4 v + \sin^2 v \cos^2 v\right) \text{and}$$

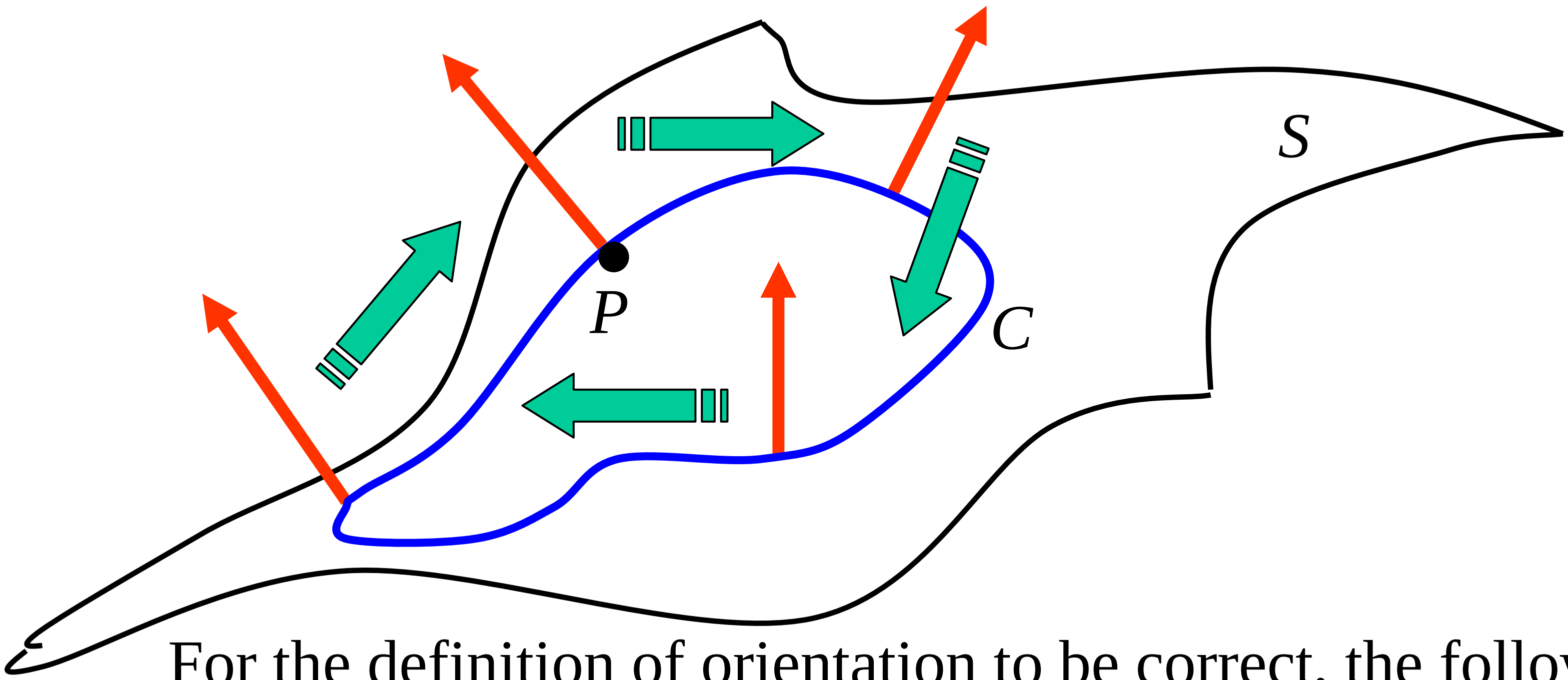
$$\sqrt{A^2 + B^2 + C^2} = r^2 \sin v$$

$$|S| = 8r^2 \int_{M'} \sin v\, du\, dv = 8r^2 \int_0^{\pi/2} du \int_0^{\pi/2} \sin v\, dv = 8r^2 \frac{\pi}{2}(1) = 4\pi r^2$$

Orientating a surface



To orientate a surface means to say which side is the „upper one“. We do this by orienting the normals. This then makes it possible to deter- mine the orientation of closed curves. (right-handed and left-handed threads).



For the definition of orientation to be correct, the following condition must be true for any closed curve C in S:

(*) After moving the normal continuously along C and returning to the starting point P, it must still have the same direction.

Orientable surfaces

3D-surfaces for which condition (*) is satisfied are called orientable.

There are 3D surfaces that do not meet this condition and are not orientable.

An example of a non-orientable 3D-surface is the ***Möbius band***

$$x = \cos u - v \sin u \sin \frac{u}{2}$$

$$y = \sin u + v \cos u \sin \frac{u}{2} \qquad u \in [0, 2\pi), v \in [-0.1, 0.1]$$

$$z = v \cos \frac{u}{2}$$