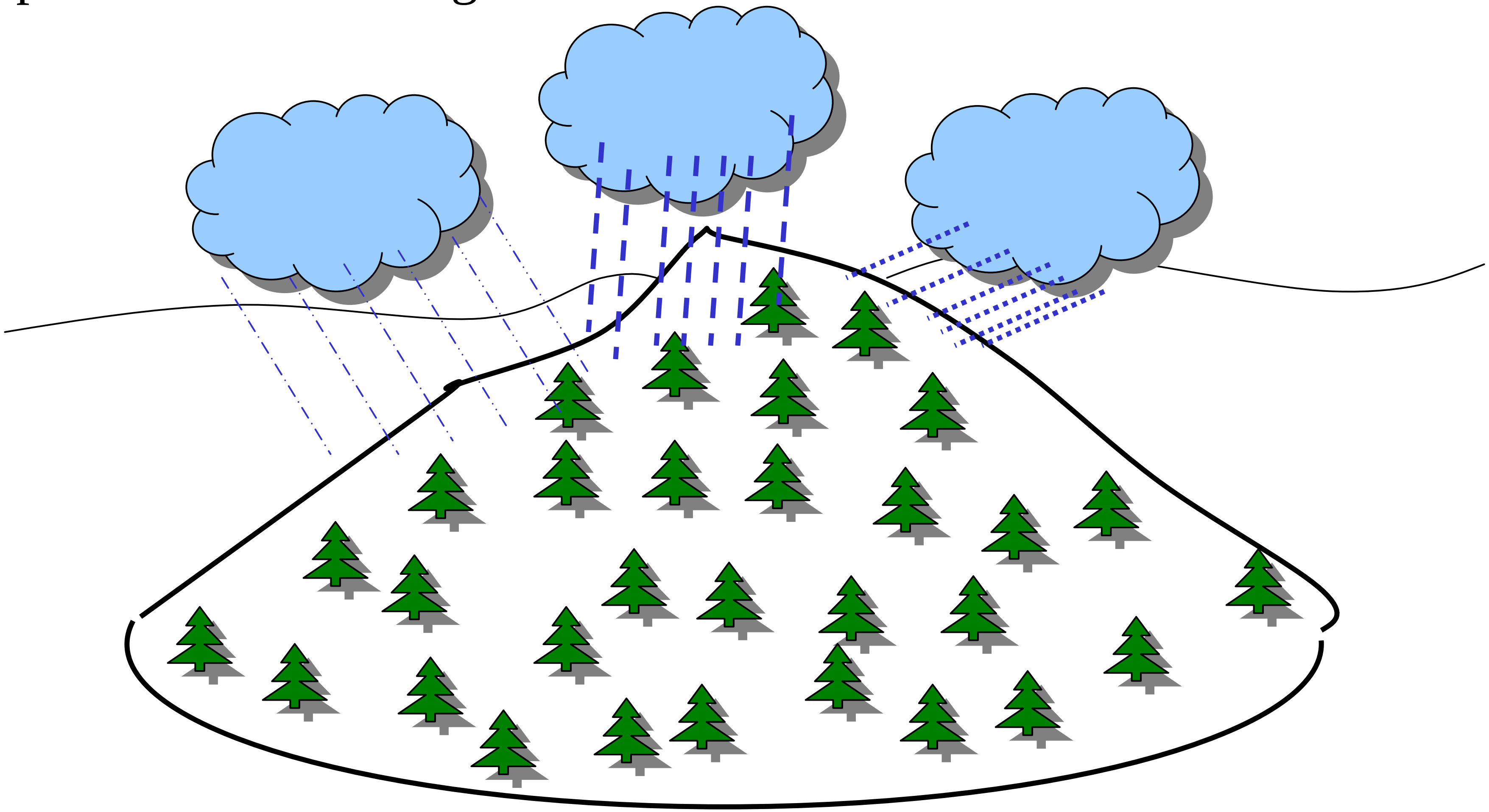
# Surface integral in a scalar field

Calculate the total amount of rainfall on a hill which has approximately the form of a cone if the precipitation rate is linearly proportionate to the height above the sea level.

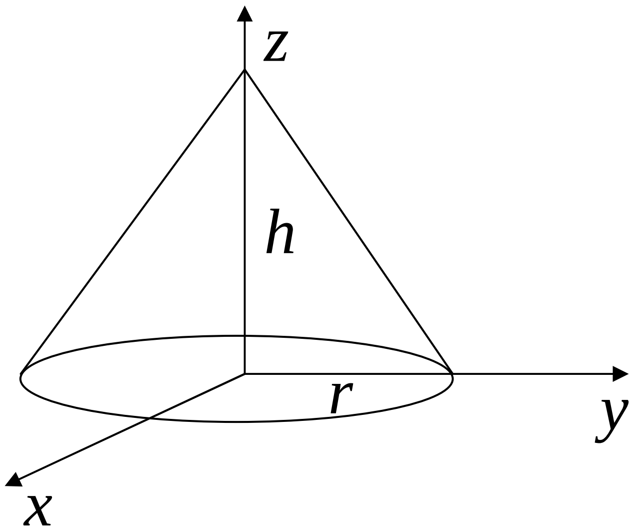


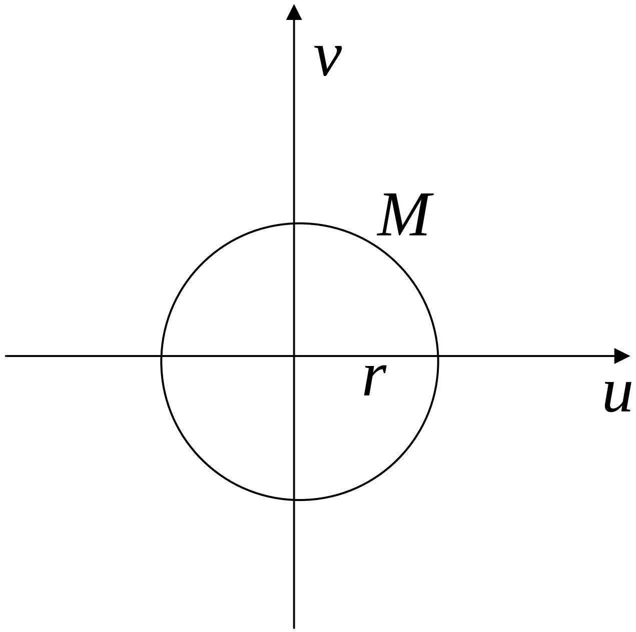
## Mathematical model:

The hill is ideally a cone and, with a height *h* and radius *r*, it may have the following parametric equations:

$$x = u,\ y = v,\ z = h - \frac{\sqrt{u^2 + v_2}}{r},\ [u, v] \in M$$

where *M* is a circle with a radius of *r* and the centre at the origin.



The precipitation rate in millimetres can be expressed as

$$p(x, y, y) = c\,z$$

where $c$ is an appropriate constant.

To calculate the total amount of rainfall $R$ fallen on the hill, we have to "add up" the precipitation rate $p(x, y, y)$ over the whole hill surface area $H$.

We will call this quantity the surface integral of $p(x, y, y)$ with respect to $H$ and denote by

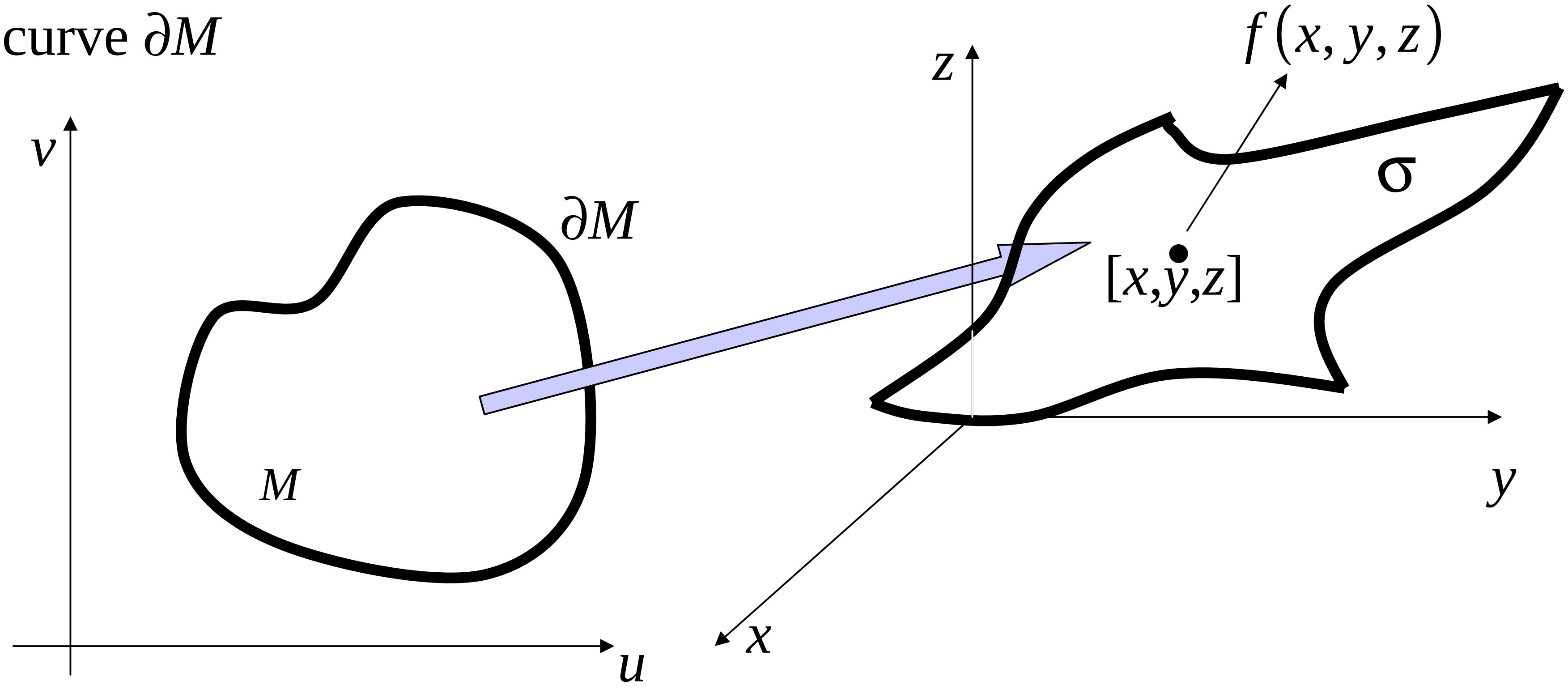
$$R = \iint_H p(x, y, z)\,dS$$

$dS$ is sometimes called the element of surface

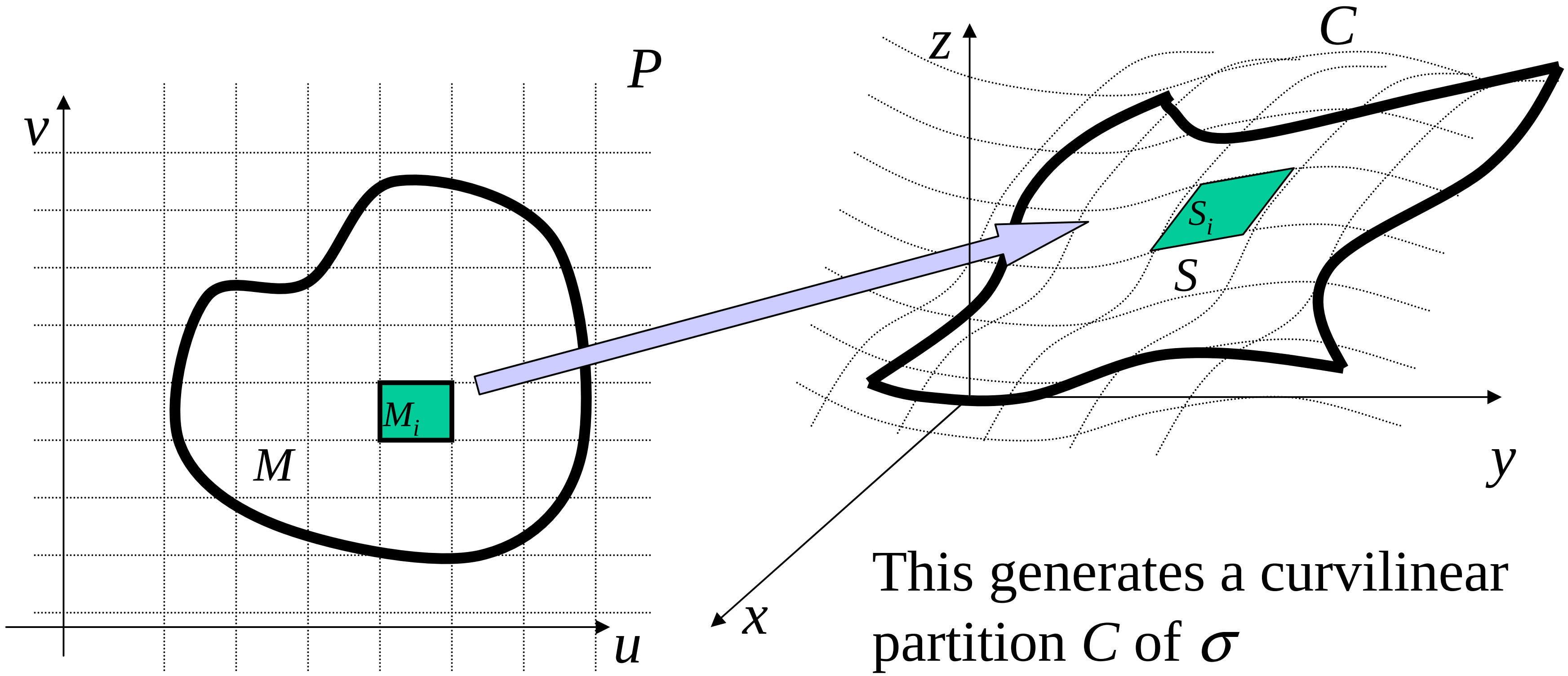
# Exact definition

Let us further assume that a continuous function $f(x,y,z)$ is defined on a a simple surface $\sigma$: $x=\varphi(u,v),\ y=\psi(u,v),\ z=\chi(u,v),\ [u,v]\in M$

where $M$ is a contiguous planar area contained within a regular curve $\partial M$



To define exactly the surface integral of $f(x,y,z)$ with respect to σ, let us first introduce a partition $P$ with a norm $N(P)$ (the area of the largest rectangle.




This generates a curvilinear partition $C$ of $\sigma$

On each $S_i \in C$ let us select an arbitrary point $[x_i, y_i, z_i]$ and calculate $|S_i| \; f(x_i, y_i, z_i)$ where $|S_i|$ is the area of $S_i$.

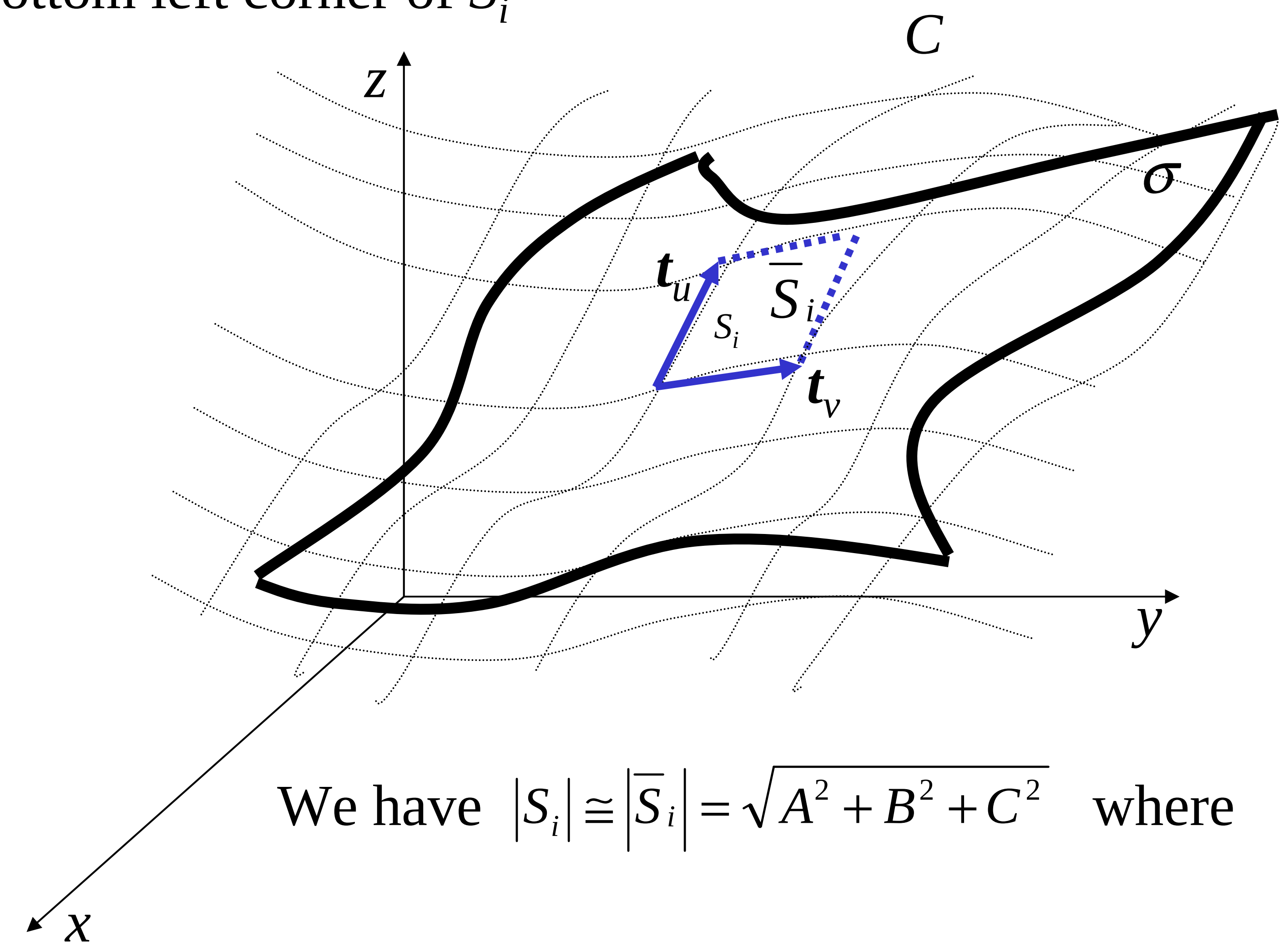
We can now define the integral sum

$$I(M, P, \sigma, Q, f) = \sum_{S_i \in Q} f(x_i, y_i, z_i)|S_i|$$

If $\lim\limits_{N(P) \to 0} I(M, P, \sigma, Q, f)$ exists, we define the surface integral of $f(x,y,z)$ with respect to $S$ as:

$$\iint_{\sigma} f(x, y, z)\, dS = \lim_{N(P) \to 0} I(M, P, \sigma, Q, f)$$

To calculate a surface integral, we replace each $S_i$ by the parallelogram $\overline{S}_i$ defined by the tangent vectors $\vec{t}_u, \vec{t}_v$ at the bottom left corner of $S_i$




$$A = \begin{vmatrix} \psi_u & \chi_u \\ \psi_v & \chi_v \end{vmatrix}$$

$$B = \begin{vmatrix} \chi_u & \varphi_u \\ \chi_v & \varphi_v \end{vmatrix}$$

$$C = \begin{vmatrix} \varphi_u & \psi_u \\ \varphi_v & \psi_v \end{vmatrix}$$

We have $|S_i| \cong |\overline{S}_i| = \sqrt{A^2 + B^2 + C^2}$ where

Then we can write

$$\iint_{\sigma} f(x,y,z)\,dS = \lim_{N(P)\to 0} I(M,P,\sigma,Q,f) = \lim_{N(P)\to 0} \sum_{S_i\in C} f(x_i,y_i,z_i)|S_i| =$$

$$= \lim_{N(P)\to 0} \sum_{S_i\in C} f(x_i,y_i,z_i)|S_i| = \lim_{N(P)\to 0} \sum_{S_i\in Q} f(x_i,y_i,z_i)\sqrt{A^2+B^2+C^2} =$$

$$= \int_{M} f(\varphi(u,v),\psi(u,v),\chi(u,v))\sqrt{A^2+B^2+C^2}\,du\,dv$$

So the formula for calculating a surface integral is:

$$\boxed{\iint_{\sigma} f(x,y,z)\,dS = \int_{M} f(\varphi(u,v),\psi(u,v),\chi(u,v))\sqrt{A^2+B^2+C^2}\,du\,dv}$$

## **Example**

Let us calculate the motivating example. We will obtain the following surface integral:

$$\iint_H c\,z\,dS \text{ where } H\text{: } x=u,\ y=v,\ z=h-\frac{\sqrt{u^2+v_2}}{r},\ [u,v]\in M$$

$$M: x=\rho\cos\varphi,\ y=\rho\sin\varphi,\ [\rho,\varphi]\in[0,r]\times[0,2\pi]$$

The tangent vectors are:

$$\boldsymbol{t}_u = 1\boldsymbol{i}+0\boldsymbol{j}+\frac{-u}{r\sqrt{u^2+v^2}}\boldsymbol{k}, \quad \boldsymbol{t}_v = 0\boldsymbol{i}+1\boldsymbol{j}+\frac{-v}{r\sqrt{u^2+v^2}}\boldsymbol{k}$$

and so we have:

$$A=\frac{u}{r\sqrt{u^2+v^2}}, \quad B=\frac{v}{r\sqrt{u^2+v^2}}, \quad C=1$$

This yields

$$\sqrt{A^2+B^2+C^2}=\sqrt{\frac{u^2}{r^2\left(u^2+v^2\right)}+\frac{v^2}{r^2\left(u^2+v^2\right)}+1}=\frac{\sqrt{1+r^2}}{r}$$

and we get the following double integral:

$$R=\iint_M\left(h-\frac{\sqrt{u^2+v^2}}{r}\right)\frac{\sqrt{1+r^2}}{r}\,du\,dv$$

After some simplification, we can write

$$R=\frac{\sqrt{1+r^2}}{r}\left(h\iint_M\,du\,dv-\frac{1}{r}\iint_M\sqrt{\left(u^2+v^2\right)}\,du\,dv\right)$$

Calculating, we obtain

$$R = \frac{\sqrt{1+r^2}}{r}\left( h\pi r^2 - \frac{1}{r}\int_0^{2\pi} d\varphi \int_0^r \rho^2 d\rho \right)$$

where in the second integral we used the transformation

$$u = \rho\cos\varphi, v = \rho\sin\varphi, \quad [\rho,\varphi] \in [0,r]\times[0,2\pi].$$

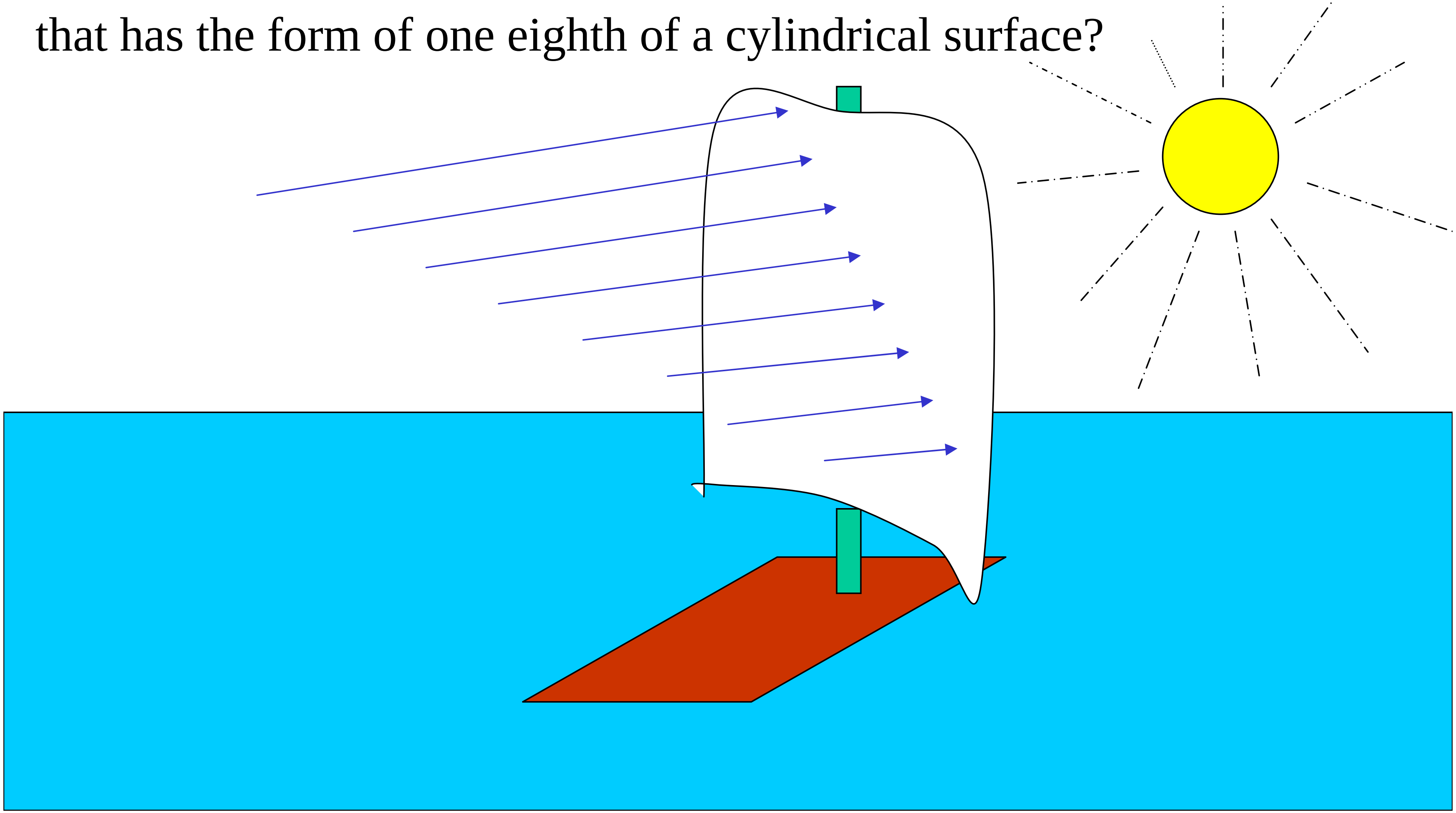
Completing the calculation, we have

$$R = \frac{\sqrt{1+r^2}}{r}\left( h\pi r^2 - \frac{1}{r} 2\pi \frac{r^3}{3} \right) = \sqrt{1+r^2}\left( h\pi r - 2\pi \frac{r}{3} \right) =$$

$$= \pi r\sqrt{1+r^2}\,\frac{3h-2r}{3}$$

## Surface integral in a vector field

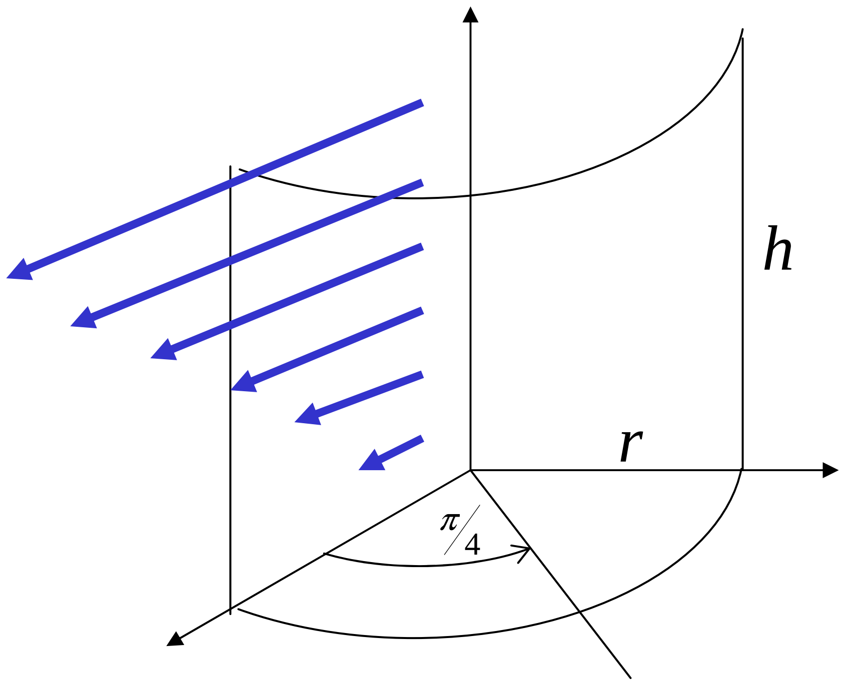
What is the force of wind blowing at an angle of 45$^o$ with force constantly increasing as we rise from the sea level driving a raft with a sail that has the form of one eighth of a cylindrical surface?



## Mathematical model

The sail is ideally an eighth of a cylindrical surface with a height $h$ and radius $r$ situated, say, in the first octant as shown in the figure below. Then it has the equations

$$x = r\cos u,\ y = r\sin u,\ z = v, \quad [u,v] \in \left[0, \tfrac{\pi}{4}\right] \times [0,h]$$



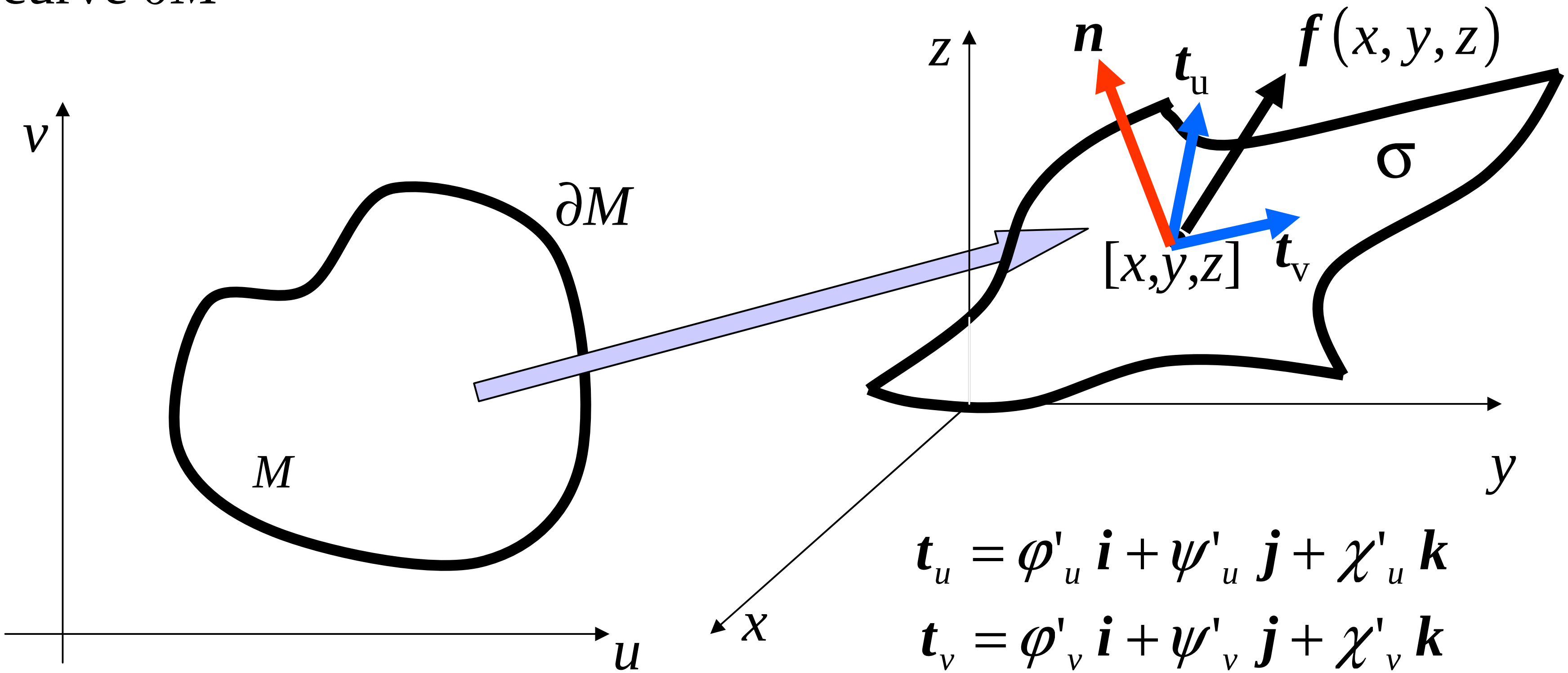

The vector field of the wind can be modeled as

$$\vec{w} = c\,z\,\vec{i} + 0\,\vec{j} + 0\,\vec{k}$$

In physics, the force with which the vector field $\vec{w} = cz\vec{i} + 0\vec{j} + 0\vec{k}$ affects a surface $S$ at a point $[x,y,z]$ is the scalar product of $\vec{w}$ with the normal $\vec{n}$ at $[x,y,z]$. Thus if, for example, the wind blew parallel to the tangent plane at $[x,y,z]$, the resulting force would be zero. All we have to do to calculate the total force of the wind acting upon the sail is "add up" the products $\vec{w} \times \vec{n}$ over all the sail surface. This operation in the case of continuous functions is called the surface integral of the second type of the vector field over the surface $S$ and in our caser we would use the following notation: $\iint\limits_S cz\,dy\,dz + 0\,dx\,dz + 0\,dx\,dy$

**Definition**

Let a continuous vector field $\vec{f}(x,y,z) = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ be defined on a a simple surface $\sigma$: $x=\varphi(u,v),\ y=\psi(u,v),\ z=\chi(u,v),\ [u,v]\in M$ where $M$ is a contiguous planar area contained within a regular curve $\partial M$

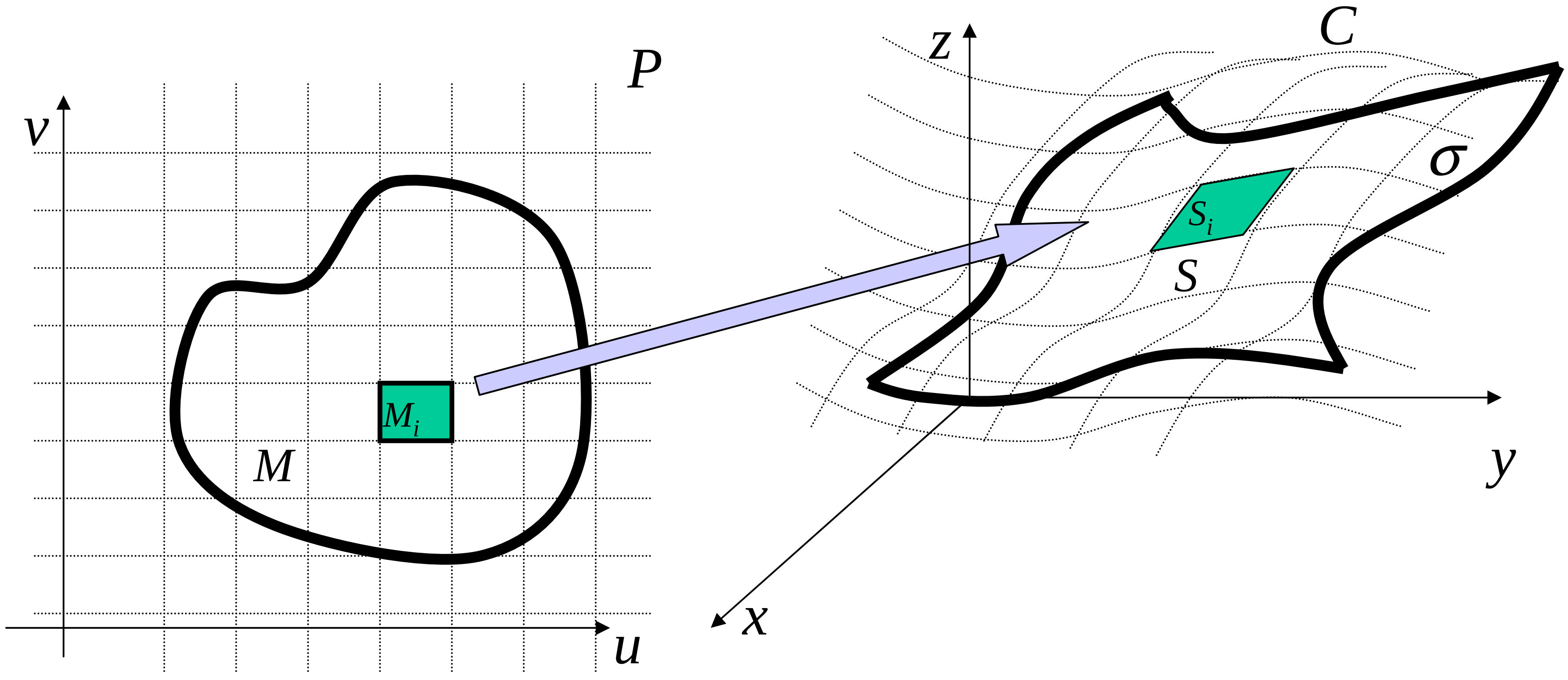



$$\boldsymbol{t}_u = \varphi'_u\,\boldsymbol{i} + \psi'_u\,\boldsymbol{j} + \chi'_u\,\boldsymbol{k}$$

$$\boldsymbol{t}_v = \varphi'_v\,\boldsymbol{i} + \psi'_v\,\boldsymbol{j} + \chi'_v\,\boldsymbol{k}$$

$$\boldsymbol{n} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$$

Let us again introduce a partition $P$ with a norm $N(P)$ with the resulting curvilinear partition $C$ of σ.



On each $S_i \in Q$ let us select an arbitrary point $[x_i, y_i, z_i]$ and calculate $\vec{n} \times \vec{f}$ at this point.

We can now define the integral sum

$$I(\sigma, Q, f) = \sum_{S_i \in Q} f_1(x_i, y_i, z_i)A + f_2(x_i, y_i, z_i)B + f_1(x_i, y_i, z_i)C$$

If $\lim\limits_{N(P) \to 0} I(M, P, \sigma, Q, f) = I$ exists, we define its value as the surface integral of $f(x,y,z)$ with respect to $S$. Formally:

$$\boxed{\iint\limits_{\sigma} f_1(x, y, z)dydz + f_2(x, y, z)dxdz + f_3(x, y, z)dxdy = I}$$

Sometimes the following vector notation is used:

$$\iint_{\sigma} f_1(x,y,z)\,dydz + f_2(x,y,z)\,dxdz + f_3(x,y,z)\,dxdy = \iint_{\sigma} \vec{f}(x,y,z)\,d\vec{S}$$

The formula for calculating a second-type surface integral of the vector field $f(x,y,z)$ can be derived in a way similar to that for the surface integral of the first type

$$\boxed{\iint_{\sigma} \boldsymbol{f}(x,y,z)\,d\boldsymbol{S} = \varepsilon \int_{M} f_1(\varphi,\psi,\chi)A + f_2(\varphi,\psi,\chi)B + f_3(\varphi,\psi,\chi)C\,dudv}$$

where ε is either 1 or -1 if depending on whether σ is oriented in correspondence with the parametric equations or not.

**Example**

We will again calculate the motivating example. Here the equat-

ions of the surface, that is, the sail $S$ are

$$x = r\cos u,\ y = r\sin u,\ z = v, \quad [u,v] \in \left[0, \pi/4\right] \times [0,h]$$

and the vector field of the blowing wind is given as

$$\vec{w} = c\,z\,\vec{i} + 0\,\vec{j} + 0\,\vec{k}$$

Let us first calculate $\vec{n}$

$$\vec{n} = \left(-r\sin u\,\vec{i} + r\cos u\,\vec{j} + 0\,\vec{k}\right) \times \left(0\vec{i} + 0\,\vec{j} + 1\vec{k}\right) =$$

$$= r\cos u\,\vec{i} + r\sin u\,\vec{j} + 0\,\vec{k}$$

We see that, for $v \in [0,\pi/2]$, $z \in [0,h]$ we have $\vec{w} \cdot \vec{n}\; czr\cos u \geq 0$

and so, for practical reasons, we will orientate the sail in correspondence with the parametric equations since we want the force of the wind to be positive. Then we can write:

$$\iint_S cz\, dydz + 0\, dxdz + 0\, dxdy = \iint_M cvr\cos u\, du\, dv =$$

$$= cr\int_0^{\pi/2} \cos u\, du \int_0^h v\, dv = \frac{crh^2}{2}$$